Similarity between Dirac and Maxwell Equations

Doron Kwiat*

17/2 Erez Street, Mazkeret Batya, Israel

*****Corresponding Author:** *Doron Kwiat, 17/2 Erez Street, Mazkeret Batya, Israel,*
Email: doron@orange.net.il

ABSTRACT

Assuming a charged electron, with a ring current distribution, the resulting Maxwell equations for the charged electron resembles Dirac equation. Thus, a connection is made between Dirac's electron field and Maxwell's electromagnetic field. Furthermore, based on the assumption of a ring shaped electron, estimation is made on the electron's radius.

Keywords: *Dirac equation; Maxwell equation; Ring-like Electrons; Electron radius*

INTRODUCTION

Ever since its discovery, the structure of the electron has been discussed by many. Many have tried to reach an explanation of its internal structure and its radius.

It is believed today that the electron as a Electron is a one of the elementary particles which cannot be divided into smaller particles. Thus the electron and the positron are the smallest massive Electrons. They have a unit charge e and spin-half and mass m.

It was Dirac[1] who described the fermion in the relativistic Dirac equation.

It is now clear that all massive elementary particles must be charged fermions with half-spin multiplications.

Much earlier, the electromagnetic field was described by Maxwell and the two fields, the fermionic field and the electromagnetic field were considered separately.

The internal structure of the electron have been the subject of many articles, from Max Born[2] who wrote in 1933: "The attempts to combine Maxwell's equations with the quantum theory (Pauli, Heisenberg, Dirac) have not succeeded. One can see that the failure does not lie on the side of the quantum theory, but on the side of the field equations, which do not account for the existence of a radius of the electron (or its finite energy=mass) ".

The classical electron radius is given by

$$R_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc^2} \approx 2.8 \; x10^{-15} \; [m]$$

Which is impossible, as it is much larger than the proton radius, which was recently [3], based on experimental electron-proton scattering, estimated to be?

$$R_P \approx 0.87 \; x10^{-15} \; [m]$$

From the close agreement of experimental and theoretical g-values[4] a new value for the electron radius $R_e < 10^{-23}$ [m] may be extracted

It was discussed earlier and stated there[5], that Maxwell's equation cannot be put into a spinor form that is equivalent to Dirac's equation. First of all, the spinor ψ in the representation of the electromagnetic field bivector depends on only three independent complex components whereas the Dirac spinor depends on four. Second, Dirac's equation implies a complex structure specific to spin 1/2 particles that has no counterpart in Maxwell's equation. This complex structure makes fermions essentially different from bosons and therefore insures, that there is no physically meaningful way to transform Maxwell's and Dirac's equations into each other.

More ecently[6]a matrix formulation of the Maxwell field equations was presented employing an 8-by-8 matrix operator, referred to then as the spacetime operator. The primary purpose of that was to employ the methods of matrix calculus to determine electromagnetic fields for arbitrary charge and current distributions. Four-dimensional Fourier transforms, transfer functions, and the convolution theorem were employed. New

matrix formulations of the classical electromagnetic (EM) Maxwell field equations and the quantum mechanical (QM) relativistic Dirac equations were presented and referred to as the Maxwell and Dirac spacetime matrix equations, respectively.

From the Dirac spacetime matrix equation they were able to form four relativistic vector equations which resemble the four Maxwell vector equations.

An extended model of the electron in general relativity was described7, where a classical model of the spinning electron was proposed in which the electron is regarded as a charged rotating shell endowed with surface tension. The shell is the surface of an oblate ellipsoid of revolution having a minor axis equal to the classical electron radius and a focal distance of the order of the corresponding Compton wavelength. This surface is undergoing rigid rotation with its equator at a velocity almost equal to the velocity of light.

Recently8, a formulation of quantum mechanics without use of complex numbers is shown to be equivalent to the complex formulation. With real wave functions formulation, the Dirac equation is presented as 4 real wave functions, solving 4 KG equations of massive point-like particles. These 4 solutions represent 4 states of a fermion.

In a recent work9, the internal structure of a charged finite size electron was examined.

Based on Maxwell's equations, the 4-vector internal electromagnetic field A^μ of the charged electron was investigated. The solution gives 4 non-homogeneous KG equations.

Under the assumption of a ring-shaped current distribution, the Maxwell equations for the electron internal vector field have the same form as the Dirac equations.

Dirac Equation in a Real Vector Space

The relativistic Dirac equation, describing a free Fermion of mass m is:

$$\left(i\hbar\gamma^\mu\partial_\mu - mc\right)\Psi = 0$$
(1)

Separate the complex wave function Ψ into its real and imaginary parts[8]

$$\Psi = \begin{pmatrix} \Psi_r \\ i\Psi_i \end{pmatrix} \qquad (2)$$

and insert them into the Dirac equation, decomposing Ψ into $\Psi_r + i\Psi_i$ where Ψ_r and Ψ_i are real, the Dirac equation becomes

$$\left(i\hbar\{\gamma^0\partial_t + \gamma^1\partial_x + \gamma^3\partial_z + i\gamma'^2\partial_y\} - mc\right)\begin{pmatrix} \Psi_r \\ i\Psi_i \end{pmatrix} = 0 \qquad (3)$$

or, Which leads (after separation of real and imaginary parts) to:

$$\partial_t\Psi_i + \sigma'_y\partial_y\Psi_r = 0 \qquad (4)$$

$$\partial_t\Psi_r - \sigma'_y\partial_y\Psi_i = 0 \qquad (5)$$

$$(\sigma_x\partial_x + \sigma_z\partial_z)\Psi_i = -\frac{mc}{\hbar}\Psi_r \qquad (6)$$

$$(\sigma_x\partial_x + \sigma_z\partial_z)\Psi_r = -\frac{mc}{\hbar}\Psi_i \qquad (7)$$

Where by definition, $\sigma_y = i\sigma'_y$ with $\sigma'_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Since both Ψ_r and Ψ_i are made of 2 real components each, there are apparently 4 constituents of the Dirac particle, denoted by, Ψ_A, Ψ_B, Ψ_C and Ψ_D.

These 4 components are: $\Psi_r = \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}$ and $\Psi_i = \begin{pmatrix} \Psi_C \\ \Psi_D \end{pmatrix}$

Eqs.4-5 can be then formulated as follows:

$$\partial_t\Psi_A + \partial_y\Psi_D = 0 \qquad (8)$$
$$\partial_t\Psi_B - \partial_y\Psi_C = 0 \qquad (9)$$
$$\partial_t\Psi_C - \partial_y\Psi_B = 0 \qquad (10)$$
$$\partial_t\Psi_D + \partial_y\Psi_A = 0 \qquad (11)$$

While Eqs. 6-7 yield:

$$\partial_x\Psi_D - \partial_z\Psi_C = -\frac{mc}{\hbar}\Psi_A \qquad (12)$$
$$\partial_x\Psi_C - \partial_z\Psi_D = -\frac{mc}{\hbar}\Psi_B \qquad (13)$$
$$\partial_x\Psi_B + \partial_z\Psi_A = -\frac{mc}{\hbar}\Psi_C \qquad (14)$$
$$\partial_x\Psi_A - \partial_z\Psi_B = -\frac{mc}{\hbar}\Psi_D \qquad (15)$$

These equations hint to interactions between the possible states of the electron.

Applying ∂_x and ∂_t to Eqs.8-15 results in

$$\left(\partial_t^2 + \partial_y^2\right)\Psi_i = 0 \quad \text{for i= A,C} \qquad (16)$$
$$\left(\partial_t^2 - \partial_y^2\right)\Psi_i = 0 \quad \text{for i= B,D} \qquad (17)$$

This can be combined by subtraction/addition to give:

$$\left(\frac{\partial^2}{c^2\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right)\Psi_i = 0 \text{ for i= A,C} \qquad (18)$$

$$\left(\frac{\partial^2}{c^2\partial t^2} + \nabla^2 - \left(\frac{mc}{\hbar}\right)^2\right)\Psi_i = 0 \text{ for i= B,D} \qquad (19)$$

We thus see that Dirac equation becomes 4 real waves' equations. Each component satisfies Klein-Gordon equation. Since the solutions to the Klein-Gordon equations are well known, we can conclude and say, that the Dirac relativistic Fermion is actually a 4entities equation. These 4entities are of same mass m.

Similarity between Dirac and Maxwell Equations

$$\frac{\partial^2 \psi_A}{c^2 \partial t^2} - \nabla^2 \psi_A = -\left(\frac{mc}{\hbar}\right)^2 \psi_A \qquad (20)$$

$$\frac{\partial^2 \psi_C}{c^2 \partial t^2} - \nabla^2 \psi_C = -\left(\frac{mc}{\hbar}\right)^2 \psi_C \qquad (21)$$

$$\frac{\partial^2 \psi_B}{c^2 \partial t^2} + \nabla^2 \psi_B = +\left(\frac{mc}{\hbar}\right)^2 \psi_B \qquad (22)$$

$$\frac{\partial^2 \psi_D}{c^2 \partial t^2} + \nabla^2 \psi_D = +\left(\frac{mc}{\hbar}\right)^2 \psi_D \qquad (23)$$

3-DIMENSIONAL MIRROR FUNCTIONS

Two functions, $f_1(x, y, z)$ and $f_2(x, y, z)$ are said to be mirror functions if $f_2(x, y, z) = -f_1(x, y, z)$ for all x,y,z.

A surface of a sphere for instance represents a 3-dimensional mirror function.

Assume now $\widetilde{\psi}_B$ and $\widetilde{\psi}_D$, to be the mirror functions of ψ_B and ψ_D, respectively.

Then $\nabla^2 \widetilde{\psi}_B = -\nabla^2 \psi_B$, and $\nabla^2 \widetilde{\psi}_D = -\nabla^2 \psi_D$. In such case,

$$\frac{\partial^2 \psi_A}{c^2 \partial t^2} - \nabla^2 \psi_A = -\left(\frac{mc}{\hbar}\right)^2 \psi_A \qquad (24)$$

$$\frac{\partial^2 \psi_C}{c^2 \partial t^2} - \nabla^2 \psi_C = -\left(\frac{mc}{\hbar}\right)^2 \psi_C \qquad (25)$$

$$\frac{\partial^2 \widetilde{\psi}_B}{c^2 \partial t^2} - \nabla^2 \widetilde{\psi}_B = +\left(\frac{mc}{\hbar}\right)^2 \widetilde{\psi}_B \qquad (26)$$

$$\frac{\partial^2 \widetilde{\psi}_D}{c^2 \partial t^2} - \nabla^2 \widetilde{\psi}_D = +\left(\frac{mc}{\hbar}\right)^2 \widetilde{\psi}_D \qquad (27)$$

In other words, we have four possible KG solutions. Two solutions (A, C) with positive energy term and two solutions (B, D) with negative energy term

Maxwell Equations for a Ring Current

Maxwell's equations and the Lorentz force law are extraordinarily successful at explaining and predicting a variety of phenomena; however they are not exact, but a classical limit of quantum electrodynamics (QED).

Some observed electromagnetic phenomena are incompatible with Maxwell's equations. These include photon–photon scattering and many other phenomena related to photons or virtual photons, "non classical light" and quantum entanglement of electromagnetic fields.
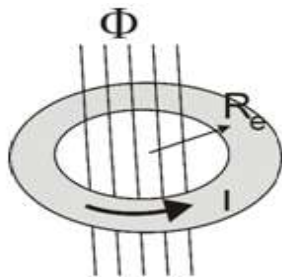

Figure1. *A ring-shaped electron of radius Re with current I and potential field ϕ*

The electromagnetic field vector

$$A^\mu = (\phi, \vec{A})$$

Satisfies Maxwell's equations:

$$\frac{\partial^2 \phi}{c^2 \partial t^2} - \nabla^2 \phi = \frac{\rho}{\varepsilon_0} \qquad (28)$$

$$\frac{\partial^2 \vec{A}}{c^2 \partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{J} \qquad (29)$$

Assume the Dirac charged electron has an internal current density[9] j^μ and hence an internal vector potential A^μ.

This particle has an internal electric charge distribution ρ, which is responsible for an internal current $\vec{J}$.

Because of current conservation and the definition of the vector potentials one has

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} = 0 \qquad (30)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \qquad (31)$$

Suppose now that the current density is proportional to the vector potential:

$$\rho = k_1 \phi \qquad (32)$$

$$\vec{J} = k_2 \vec{A} \qquad (33)$$

(Which, for example, is the case of the vector potential and current density for a current in a ring)?

Then

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = k_1 \frac{\partial \phi}{\partial t} + k_2 \nabla \cdot \vec{A} = k_2 \left(\frac{k_1}{k_2}\frac{\partial \phi}{\partial t} + \nabla \cdot A\right) = 0 \qquad (34)$$

Provided that $k_1/k_2 = 1/c^2$, then if Eq. 30 is true, then so is Eq. 31 and vice versa, Eq. 32 will lead to Eq. 30. Then, the conservation equations (Eqs. 30, 31) are always satisfied. Thus, the proportionality conditions (Eqs. 32,33), are enough to make the charge-current conservation will force the gauge condition Eq. 30, and the gauge condition Eq. 30, will force charge current conservation, Eq. 31.

Then, from Eqs. 28, 29one obtains:

$$\frac{\partial^2 \phi}{c^2 \partial t^2} - \nabla^2 \phi = k_1 \phi \qquad (35)$$

$$\frac{\partial^2 \vec{A}}{c^2 \partial t^2} - \nabla^2 \vec{A} = k_2 \vec{A} \qquad (36)$$

And $k_1/k_2 = \varepsilon_0 \mu_0 = 1/c^2$ \qquad (37)

Now, based on the fact that the units of $[k_1] = [meter^{-1}]$, it will be assumed ($\frac{mc}{\hbar}$ has dimensions of meter^{-1}) that

$$\rho = \varepsilon_0 \left(\frac{mc}{\hbar}\right)^2 \phi \qquad (38)$$

Similarity between Dirac and Maxwell Equations

$$\vec{J} = \frac{1}{\mu_0}\left(\frac{mc}{\hbar}\right)^2 \vec{A} \tag{39}$$

Thus, $k_1 = \varepsilon_0\left(\frac{mc}{\hbar}\right)^2$ and $k_2 = \frac{1}{\mu_0}\left(\frac{mc}{\hbar}\right)^2$

When inserted into Maxwell Eqs. 28, 29, they become of same form as the Dirac 4-componentsEqs. 24-27:

$$\frac{\partial^2\phi}{c^2\partial t^2} - \nabla^2\phi = \left(\frac{mc}{\hbar}\right)^2\phi \tag{40}$$

$$\frac{\partial^2 A_1}{c^2\partial t^2} - \nabla^2 A_x = \left(\frac{mc}{\hbar}\right)^2 A_x \tag{41}$$

$$\frac{\partial^2 A_2}{c^2\partial t^2} - \nabla^2 A_y = \left(\frac{mc}{\hbar}\right)^2 A_y \tag{42}$$

$$\frac{\partial^2 A_3}{c^2\partial t^2} - \nabla^2 A_z = \left(\frac{mc}{\hbar}\right)^2 A_z \tag{43}$$

In other words, the four Dirac fields $\psi_A, \widetilde{\psi}_B, \psi_C, \widetilde{\psi}_D$ are equivalent to the electromagnetic vector potential fields, ϕ, A_x, A_y, A_z provided that the current densities ρ, J_x, J_y, J_z are related to the electromagnetic fields ϕ, A_x, A_y, A_z by Eqs. 37,38 through the constant $\left(\frac{mc}{\hbar}\right)^2$.

So, if one identifies the four Dirac fields $(\psi_A, \widetilde{\psi}_B, \psi_C, \widetilde{\psi}_D)$ with the vector potential (ϕ, A_x, A_y, A_z), Eqs. 39-42, become identical with Eqs. 24-27.

Note though, that in order to keep with signs, as required byEqs24 -27, the currents $\tilde{J}_x$ and $\tilde{J}_z$ must also be symmetric with J_x and J_z about their spatial coordinates (mirrors), so that their spatial derivatives change signs accordingly.

There are no external electromagnetic fields, so these are confined electromagnetic fields which create the electron and give it its mass m.

Maxwell Eqs for a ring current	Dirac Eqs
$\dfrac{\partial^2\phi}{c^2\partial t^2} - \nabla^2\phi = +\left(\dfrac{mc}{\hbar}\right)^2\phi$	$\dfrac{\partial^2\psi_A}{c^2\partial t^2} - \nabla^2\psi_A = -\left(\dfrac{mc}{\hbar}\right)^2\psi_A$
$\dfrac{\partial^2 A_1}{c^2\partial t^2} - \nabla^2 A_x = +\left(\dfrac{mc}{\hbar}\right)^2 A_x$	$\dfrac{\partial^2\psi_C}{c^2\partial t^2} - \nabla^2\psi_C = -\left(\dfrac{mc}{\hbar}\right)^2\psi_C$
$\dfrac{\partial^2 A_2}{c^2\partial t^2} - \nabla^2 A_y = +\left(\dfrac{mc}{\hbar}\right)^2 A_y$	$\dfrac{\partial^2\tilde{\psi}_B}{c^2\partial t^2} - \nabla^2\tilde{\psi}_B = +\left(\dfrac{mc}{\hbar}\right)^2\tilde{\psi}_B$
$\dfrac{\partial^2 A_3}{c^2\partial t^2} - \nabla^2 A_z = +\left(\dfrac{mc}{\hbar}\right)^2 A_z$	$\dfrac{\partial^2\tilde{\psi}_D}{c^2\partial t^2} - \nabla^2\tilde{\psi}_D = +\left(\dfrac{mc}{\hbar}\right)^2\tilde{\psi}_D$

Notice the difference in sign of the energy terms between the ψ_A, ψ_C and the ϕ, A_x equations.

One possible explanation could be that the electron may have a positive charge and a negative charge, together with two possible opposing current directions (spins). Therefore, Maxwell equations will describe opposite spins, only if we assume opposite currrents. Dirac equations contains opposite spins as a result of mirror functions.

Electron Radius

Many attempts are known to evaluate the electron's radius7-10.

The classical radius of the electron is calculated by equating the electrostatic potential energy of a sphere of charge e and radius R_e with the rest energy of the electron. It is $R_e = 2.8179 \times 10^{-15}$ m

If we compare this classical radius, with the measured radius of a proton, which is[3]$R_P \approx 1.11 \times 10^{-15}$ m, an electron has a radius 2.5 times larger than a proton. Given that a proton is around 2000 heavier, one should not take this 'classical radius' seriously.

A more realistic approach would be to take the ratio of proton/electron mass, then divide the proton's radius by the cube root of this number (because mass increases according to the cube of radius). This ratio, 1836, would set the electron's radius at approximately 12 times smaller than a proton, namely$R_e \approx 9.1 \times 10^{-17}$ m.

However, the proton is believed to consist of quarks. If these quarks occupy only a small volume within a proton, this would greatly lower its density and the electron's radius would calculate to be far smaller again, perhaps by an additional factor of 10 or 100.

A classical model of the spinning electron was proposed[10] in which the electron is regarded as a charged rotating shell endowed with surface tension. The shell is the surface of an oblate ellipsoid of revolution having a minor axis equal to the classical electron radius and a focal distance of the order of the corresponding Compton wavelength. This surface is undergoing rigid rotation with its equator at a velocity almost equal to the velocity of light. The arrangement of charges gives rise to a quadrupole electric moment, in addition to the

magnetic dipole moment of the current distribution.

In this work, the electron is assumed to have a doughnut shape, of radius R_e with a ring-current. Based on Eq. 37, the charge e of this electron will be given by:

$$e = \int_0^{2\pi} \rho R_e \, d\theta = R_e \varepsilon_0 \left(\frac{mc}{\hbar}\right)^2 \int_0^{2\pi} d\theta \, \phi \quad (44)$$

Assuming ϕ, the electromagnetic potential, to be symmetrical and constant ϕ_0, independent of θ, one obtains

$$e = 2\pi R_e \varepsilon_0 \left(\frac{mc}{\hbar}\right)^2 \phi_0 \quad\quad (45)$$

This expression creates a connection between the electron radius, its mass, and the two constants of nature c, and $\hbar$.

Calculation of this radius gives $R_e = 4.29 \times 10^{-34} \phi_0$ m, with some ϕ_0 constant value.

According to this model, the mass/radius ratio of the proton to electron (both having the same charge magnitude and assuming the same potential) would be:

$$\frac{R_P}{R_e} = \left(\frac{m_P}{m_e}\right)^2 \approx 3.46 \times 10^6$$

Thus, based on the known proton radius3 ($R_P \approx 1.11 \times 10^{-15}$ m) the estimated electron radius will be

$$R_e \approx 3.2 \times 10^{-22} \text{ m}$$

Electron Spin

According to this ring shape model, the electron will have a spin according to the direction of the ring current and the ring mass m.

The spin magnetic moment will depend on the charge ring current.

The spin magnetic moment is given by the ring area times the current I:

$$\mu_s = \pi R_e^2 I \quad\quad (46)$$

$$I = \frac{\partial \rho}{\partial t} = \varepsilon_0 \left(\frac{mc}{\hbar}\right)^2 \frac{\partial \phi}{\partial t} \quad\quad (47)$$

$$\mu_s = -\pi R^2 \varepsilon_0 \left(\frac{mc^2}{\hbar}\right)^2 \nabla \cdot \vec{A} \quad\quad (48)$$

Therefore, the spin of the electron will depend on the direction of the current clockwise or anticlockwise.

This model cannot hold for the proton, as it is composed of quarks.

Experimentally we know that the spin direction is dictated by the direction of the externally applied magnetic field. This means that the electron will align its z-direction according to and with the direction of the applied magnetic field.

CONCLUSIONS

Based on real wave functions description of Dirac's equation, similarity of Dirac with the Maxwell equations was shown.

It was then assumed that the charged electron is not a point like particle, but rather a finite doughnut shaped particle of radius R_e, carrying a ring current.

Its internal charge dynamics creates a vector potential A^μ. Under certain assumptions on the proportionality of the vector potential to the current density, the Dirac equation wave functions are similar to the internal electromagnetic vector potential.

Based on this model, the electron radius is directly connected to the Planck's and speed of light universal constants, together with the permittivity electric constant and the electron mass.

The electron radius is then predicted in comparison to the known proton radius, and is estimated at 3.2×10^{-22} m.

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