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The Equation of Motion in Rindler Space

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Abstract

The equation of motion of a massive particle in Rindler space or equivalently in an uniform gravitational field has been studied in both classical and quantum mechanical scenarios. The classical geodesics of motion of the particle and the wave function for the quantum mechanical equation satisfied by the particle are obtained. It has been observed that it is equivalent to the problem of free fall of the particle at the center. Unlike the conventional scenario where the fall occurs at the origin, which is a singularity, here it takes place at the point which is at a finite distance from the origin (singularity). Further, the classical trajectories are not a perihelion type. These are closed type. The massive particle returns to its original position if disturbed from the equilibrium position (having minimum energy).

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Introduction

It is well known that the Lorentz transformations of space time coordinates are between the inertial frame of references [1], whereas the Rindler transformations are between an inertial frame and the frame undergoing uniform accelerated motion [2-10]. The space here is called the Rindler space, exactly like the Minkowski space in the Lorentz transformation scenario. Further, the Rindler space is locally flat, whereas the flatness of Minkowski space is global in nature. Now according to the principle of equivalence a frame undergoing accelerated motion in absence of gravity is equivalent to a frame at rest in presence of gravitational field. The strength of the gravitational field is exactly equal to the magnitude of the acceleration of the moving frame. Therefore the Rindler space may be considered to be a space associated with a frame at rest in presence of a uniform gravitational field. The Rindler space-time transformations in natural unit ($c = \hbar = 1$) and in 1+1 dimension are given by [7-10]

$$t = \left(\frac{1}{\alpha} + x'\right)\sinh(\alpha t') \qquad (1)$$

$$x = \left(\frac{1}{\alpha} + x'\right)\cosh(\alpha t') \qquad (2)$$

Where primed coordinates are in the non-inertial frame, whereas the unprimed are in the inertial one. The Rindler space-time transformations are therefore exactly like the Lorentz transformations but in uniformly accelerated frame. Here α is the local acceleration or local uniform gravitational field. Hence the line element in Rindler space in 1+1 –dimension is given by

$$ds^2 = -d\tau^2 = -(1+\alpha x)^2 dt^2 + dx^2 \qquad (3)$$

Where $\boldsymbol{\tau}$ is the proper time. Then one can define the classical action integral in the form [1]

$$\int_A^B d\tau = \int_A^B L\, d\boldsymbol{\lambda} \qquad (4)$$

Where λ is an affine parameter which is changing along the trajectory and having the dimension of time. Hence $L = \frac{d\tau}{d\lambda}$, the Lagrangian of the particle, explicitly given by[11-13]

$$L = \left[(1+\alpha x)^2 + \left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dx}{d\lambda}\right)^2\right]^{\frac{1}{2}} \qquad (5)$$

The aim of the present article is to investigate the classical motion as well as the quantum mechanical motion of the non-zero mass particle in Rindler space. Apparently from the title of this article it may sound that it is a very old problem, however so far our knowledge is concerned it has not been studied before. Therefore we expect that this work should be reported in some reputed journal. In this article we have obtained the classical geodesics of motion and the wave functions for the quantum mechanical equation satisfied by the particle. It has been observed that the problem is equivalent to the problem of free fall. We have also investigated this free fall problem of the particle. Although the gravitational field is considered to be uniform throughout, however the energy of the particle is minimum at the origin. As a consequence the particle will fall at the point which is at a finite distance from the origin $(x = 0)$.Now in the Rindler coordinate system, the portion $x > |t|$ of the Minkowski space is called the Rindler wedge. The second wedge $x < -|t|$can be obtained by reflection. We call the first one as right wedge and the second one the left wedge of Rindler space. The null rays act as the event horizons for Rindler observers. An observer in the right wedge cannot see any eventsin the left wedge. These two regions are casually disjoint two universes.However, exactly like the Minkowski space the past and the present can be defined and are casually connected. Therefore the actual singularity $x = -\frac{1}{\alpha}$is not an accessible to an observer in right wedge. The origin (which is not a singularity) is set at $x = 0$. To the best of knowledge this problem has not been done before.Various sections of the article are organized in the following manner. In the next section we shall develop the formalism of the classical motion of the particle in Rindler space. In section-3 we have discussed the quantum mechanical motion in Rindler space and finally in section-4 we have given the conclusion of our work.

CLASSICAL MOTION IN RINDLER SPACE

Now the well-known Euler-Lagrange equation is given by

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial L}{\partial\left(\frac{\partial q_i}{\partial\lambda}\right)}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (6)$$

Where $q_i = t$ or x. Let us first consider $q_i = t$, the universal time coordinate. Since t is cyclic coordinate, $\frac{\partial L}{\partial t} = 0$ and therefore

$$\frac{\partial t}{\partial\tau} = \frac{C}{(1+\alpha x)^2} \qquad (7)$$

Where C is a constant and to obtain this equation we have used the expression for the Lagrangian as given by equation (5). We next consider $q_i = x$, then with some little algebra, we have from the Euler-Lagrange equation

$$(1+\alpha x)\alpha\left(\frac{\partial t}{\partial\tau}\right)^2 + \frac{d^2x}{d\tau^2} = 0 \qquad (8)$$

On substituting the expression for $\frac{\partial t}{\partial\tau}$ from equation (7) we have

$$\frac{\alpha C}{(1+\alpha x)^3} + \frac{d^2x}{d\tau^2} = 0 \qquad (9)$$

(see Appendix for some outline to obtain this expression). Now changing the variable from x to $u = 1 + \alpha x$ and redefining $\tau \to C^{\frac{1}{2}}\alpha\boldsymbol{\tau}$, we have from the above equation,

$$\frac{d^2u}{d\tau^2} + \frac{1}{u^3} = 0 \qquad (10)$$

This is the classical equation of motion for the particle in Rindler space. Integrating both sides with respect to u, this equation may be transformed to the form

$$\frac{1}{2}\left(\frac{du}{d\tau}\right)^2 - \frac{1}{2u^2} = constant \qquad (11)$$

Re-writing $\boldsymbol{\tau}$ in its original form and defining $p = \frac{du}{d\tau}$ as the momentum of the particle of unit mass, we have from the above equation

$$\frac{p^2}{2} - \frac{C\alpha^2}{2u^2} = constant = E \qquad (12)$$

Where the first term on the left hand side is like the kinetic energy for a unit mass and the second term is an attractive inverse square potential, whereas the constant on the right hand side may be treated as the total energy of the particle. We further assume that the energy of the particle is negative in nature, i.e., we are considering bounded motion of the classical particle. Now expressing the above equation in the usual form, given by

$$\frac{1}{2}p^2 + V(u) = E \qquad (13)$$

Where $V(u) = -\frac{C\alpha^2}{2u^2}$, the inverse square attractive potential. Then depending on the strength $0.5C\alpha^2$ of the attractive potential $V(u)$, the particle may fall at the center($x = 0$ or $u = 1$), where the momentum becomes zero ($p = 0$, i.e., particle is in the rest condition) and in this situation $E = V(u = 1) = -0.5C\alpha^2$, the maximum depth of the potential, which is also the minimum possible value of the total energy. Since it is negative it gives the maximum binding at $x = 0$ or $u = 1$. Since α is the uniform acceleration, the parameter C may be treated as the strength of the potential $V(u)$. One can calculate the two classical turning points in case the motion is oscillatory between two extreme points, where the particle momentum becomes zero. These are at $u = u_0 = 1$, the central point in the transformed special coordinate and $u = u_1 = (0.5C|E|)^{\frac{1}{2}}\alpha$ for E negative $= -|E|$. In this case, of course the motion is periodic. However, if because of strong attractive gravitational potential at $u = u_0$ (for the large value of C) the particle falls at the center, i.e., at $x = 0$ or $u = u_0 = 1$ then the motion will no longer remain oscillatory. In fig.(1) we have shown this potential well. The coordinate x is plotted along x -axis, where the magnitude of depth of potential $|V(u)|$ is plotted along y -axis. Therefor in actual scenario this is not the potential hill but is potential well. But unlike the conventional scenario of fall at the center problem, there is no singularity at the origin $(x = 0)$, or at the center of the gravitating object[14-16]. The value of $V(u)$ is $-0.5C\alpha^2$ at the origin and $\rightarrow -1$ for large u values. Therefore unlike the conventional case, where the potential part $\rightarrow -\infty$ at the origin, here, since the minimum value of u is positive unity, therefore $V(u) \rightarrow -0.5C\alpha^2$ for $x \rightarrow 0$ or $u \rightarrow 1$. In this case the singularity is as if covered by a point which is at finite distance from the origin or the actual singularity. Here the origin has been shifted to $x = 0$ point. Therefore here the problem of fall at the origin reduces to the problem of fall at a point which is having finite positive special coordinate value.

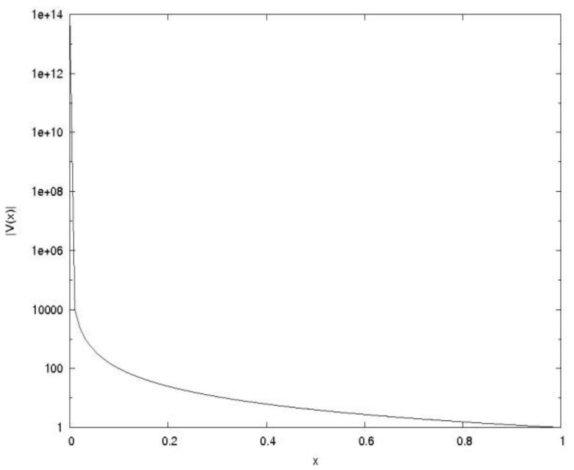

Figure1. *The variation of* $|V(x)|$ *with* x

For further investigation of this problem of fall at the center to some extent elaborately, let us get the geodesics of motion for the particle in Rindler space. We start with the equation

$$(1+\alpha x)^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 = 1 \qquad (14)$$

Hence we have after substituting for $\frac{dt}{d\tau}$ from eqn.(7)

$$\tau = \int_0^x \frac{dx}{\left[\frac{C}{(1+\alpha x)^2} - 1\right]^{\frac{1}{2}}} \qquad (15)$$

Using the changed variable $u(= 1 + \alpha x)$, we have,

$$\alpha\boldsymbol{\tau} = -\frac{C^{\frac{1}{2}}}{\alpha}\left[\left\{\mathbf{1} - \left(\frac{1+\alpha x}{C^{\frac{1}{2}}}\right)^2\right\}^{\frac{1}{2}} - \left(\mathbf{1} - \frac{1}{C}\right)^{\frac{1}{2}}\right] \qquad (16)$$

Since the uniform acceleration α of the frame is completely arbitrary, we shall study the variation of αx with $\alpha\boldsymbol{\tau}$, keeping in consideration that α remains constant within this x range and is static in nature. The above equation can also be expressed in the following more convenient form

$$\alpha x = C^{\frac{1}{2}}\left[1 - \left\{\left(1 - \frac{1}{C}\right)^{\frac{1}{2}} - \frac{\alpha\tau}{C^{\frac{1}{2}}}\right\}^2\right] - 1 \qquad (17)$$

Which will give the variation of αx with $\alpha\boldsymbol{\tau}$.

Before we obtain numerically the variation of αx with $\alpha\boldsymbol{\tau}$, we shall impose some constraints on the quantities appearing in the above equation. By inspection one can put the following restrictions: $C > 1$, and if $\alpha x > 0$, then $\alpha\boldsymbol{\tau} > 0$. Since we are measuring x along positive direction and the frame is also moving with uniform acceleration α along positive x -direrction, the second constraint is also quit obvious. In fig.(2) we have shown the variation of αx with $\alpha\boldsymbol{\tau}$ for three different C values. From the curves one may infer that the geodesics are closed in nature; returned to the initial point, i.e., at $x = 0$ or $u = 1$ (Boomerang type geodesic), where the energy of the particle is minimum.

The geodesics are not like the perihelion precession type as has been obtained in Schwarzschild geometry [17,18]. Although the gravitational field is constant throughout the region, the turning down of the particle, when it is dynamically disturbed from its equilibrium position is because of the minimum value of the energy of the particle at the origin $(x = 0)$. For the proper justification of our arguments as given above, we next consider the equation of motion of the particle under inverse square potential, given by eqn.(10). This equation has been solved numerically for the initial conditions $\frac{du}{d\tau} = 0$, i.e., the motion is assumed to be started from rest and for a number of initial positional coordinates in the scaled form given by u_0. The boundary value of u is unity, not the singular point $u = 0$, which is in the left Rindler wedge. In fig.(3) we have shown the geodesics of motion for three different u_0 values.

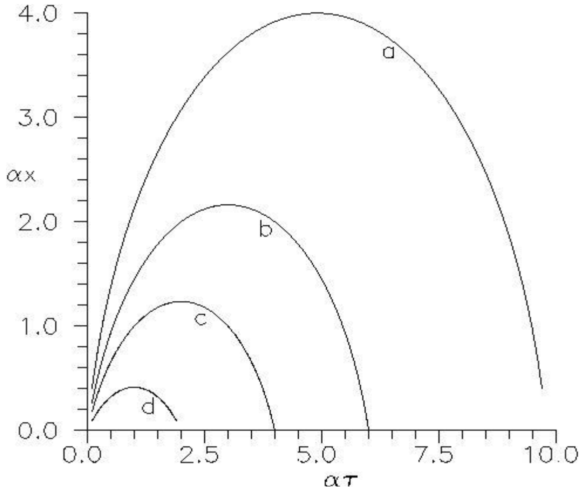

Figure2. *The geodesics of motion for various C values: curve (a) is for $C = 2$, (b) is for $C = 5$, (c) is for $C = 10$, and (d) is for $C = 25$.*

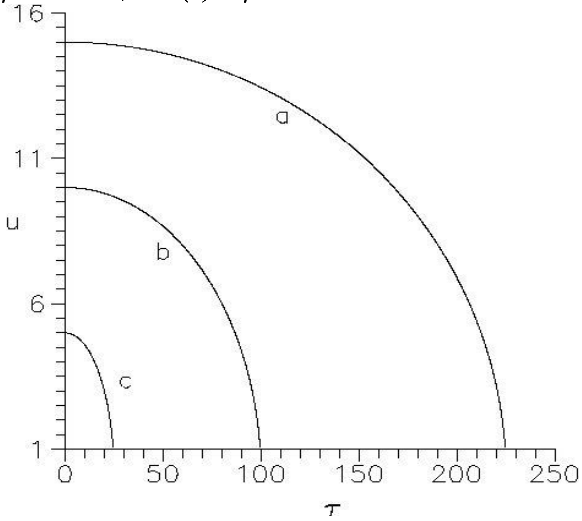

Figure3. *The geodesics of motion for various u_0 values: curve (a) is for $u_0 = 15$, (b) is for $u_0 = 10$ and (c) is for $u_0 = 5$.*

Quantum Mechanical Equation

Now to complete the study of free fall at the center in our modified formalism , let us next investigate the quantum mechanical motion of the particle in inverse square attractive potential [14,16]. We consider the bound state problem, i.e., $E < 0$. Re-substituting the exact form of **τ**, we have the eigen value equation for a unit mass particle

$$H\Psi(u) = \left(\frac{p^2}{2} - \frac{\beta}{u^2}\right)\Psi(u) = -E\Psi(u) \qquad (18)$$

Where $\beta = \frac{\alpha^2 C}{2}$. Hence we have

$$\frac{d^2\Psi}{du^2} + \frac{\gamma}{u^2}\Psi(u) - K^2\Psi(u) = 0 \qquad (19)$$

Where $\gamma = \frac{2\beta}{\hbar^2}$ and $K^2 = \frac{2E}{\hbar^2}$. Therefore in the asymptotic region, i.e., $u \to \infty, \Psi \sim e^{-Ku}$ or it goes to zero. The wave function is therefore well behaved at infinity. Next to study the nature of wave function near the point $u = 1$, we substitute $\Psi(u) = \frac{R(u)}{u}$. Then it can very easily be been shown that $R(u)$ satisfies the equation

$$u^2\frac{d^2R}{du^2} - 2u\frac{dR}{du} + (\gamma - 2 - K^2u^2)R = 0$$
(20)

Putting $Ku = \rho$ as a new variable, we have,

$$\frac{d^2R}{d\rho^2} - \frac{2}{\rho}\frac{dR}{d\rho} + \frac{\gamma-2}{\rho^2}R - R = 0 \qquad (21)$$

To investigate the nature of R near unit point we write $\rho = \rho_0$ in the denominator of the second and the third term and then put the limit $\rho_0 \to 1$. Then we have

$$\frac{d^2R}{d\rho^2} - 2\frac{dR}{d\rho} + (\gamma - 3)R = 0 \qquad (22)$$

Then for $\gamma < 4$, we have the solution

$$u(\rho) = \exp(\rho)\left[A_1 exp\left((4-\gamma)^{\frac{1}{2}}\rho\right) + \right.$$
A2exp−4−γ12ρ (23)

Whereas for $\gamma > 4$, the solution is given by

$$u(\rho) = \exp(\rho)\left[A_1 exp\left\{i(\gamma-4)^{\frac{1}{2}}\rho\right\} + \right.$$
A2exp−iγ−412ρ (24)

The solutions are standard stationary type and exist for $\rho = \rho_0 \to K$. Further, for bound state solution, i.e., for $E < 0, E_K < |V|$, where E_K and V are the kinetic energy and attractive potential respectively. The quantity $|V|$ is maximum for $\rho = \rho_0 = K$, i.e., at $x = 0$ and approaches unity as ρ increases to infinity. Therefore the negative value of E is maximum on the unit point. This indirectly indicates the "fall" on the point at $u = 1$. The basic difference with the usual solution is that the fall is not at the center ($\rho = 0$) which is covered by the unit point.

Conclusion

Therefore we may conclude that in the Rindler space the energy of the particle will be minimum at the point which is the newly defined origin or $x = 0$ or $u = 1$. This indirectly indicate that the particle will occupy this point because of the minimum valus of the energy. This is true for classical as well as quantum mechanical scenarios. The other interesting finding of this work is that unlike the perihelion type geodesics these are closed type. We call them as boomerang geodesics.

APPENDIX

In this appendix we shall give a short outline to show that $\frac{1}{L}\left[(1+\alpha x)\alpha\left(\frac{dt}{d\lambda}\right)^2+\frac{d^2x}{d\lambda^2}\right]=\frac{1}{L}\left[(1+\alpha x)\alpha\left(\frac{dt}{d\tau}\right)^2\left(\frac{d\tau}{d\lambda}\right)^2+\frac{d^2x}{d\tau^2}\frac{d^2\tau}{d\lambda^2}\right]=\frac{\alpha C}{(1+\alpha x)^3}+\frac{d^2x}{d\tau^2}=0$ (25)

We have

$$\frac{dt}{d\lambda}=\frac{dt}{d\tau}\frac{d\tau}{d\lambda}=L\frac{dt}{d\tau} \quad (26)$$

Hence

$$\left(\frac{dt}{d\tau}\right)^2=L^2\left(\frac{dt}{d\tau}\right)^2 \quad (27)$$

$$\frac{d^2x}{d\lambda^2}=\frac{d}{d\lambda}\left(\frac{dx}{d\lambda}\right)=\frac{d}{d\tau}\left(\frac{dx}{d\lambda}\right)\left(\frac{d\tau}{d\lambda}\right) \quad (28)$$

$$\frac{d^2x}{d\lambda^2}=\frac{d}{d\lambda}\left(\frac{dx}{d\tau}\right)\left(\frac{d\tau}{d\lambda}\right) \quad (29)$$

$$\frac{d^2x}{d\lambda^2}=\frac{d^2x}{d\tau^2}\left(\frac{d\tau}{d\lambda}\right)^2=L^2\frac{d^2x}{d\lambda^2} \quad (30)$$

Hence we have the above equality after putting, $\frac{dt}{d\lambda}=\frac{C}{(1+\alpha x)^2}$

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