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Solving Differential Equations by Partial Integrating Factors

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ABSTRACT

The field of Wave Theory is quite substantial. This field is considered something where an imagination is required to understand. We have unknown questions in this field that need research conducted; contributing to progression of partial differential equations and mathematics in general. The goal of this paper is to develop new concepts in Wave Theory; specifically towards the Navier-Stokes equation. Part of this concept development is showing velocity and pressure functions to exist before moving forward in the field. Imagine a theory existing prior to its manifestation. The missing concepts may have been found. Lets begin by recalling a concept from ordinary differential equations.

Keywords: *Partial Differential Equations; Differential Equations; Integrating Factors; Partial Integrating Factors; Partial Integration by Parts; Partial Integration with Exponentials*

INTRODUCTION

Recall computing integrating factors to solve ordinary differential equations. I will yield an example of integrating factors as from any known textbook on differential equations. We are going to find a general solution of the ordinary differential equation $\frac{dy}{dx}+\frac{y}{x}=x^{-2}$.

To find the general solution to this equation, locating the integrating factor is required. Let the integrating factor be

$$I=e^{\int Pdx}=e^{\int \frac{1}{x}dx}=e^{lnx}=x.$$

Multiplying the equation by our integrating factor will give us

$$x\frac{dy}{dx}+(x)\frac{y}{x}=x*x^{-2}$$

$$x\frac{dy}{dx}+y=\frac{1}{x}$$

$$xy=lnx+C$$

$$y=\frac{lnx+C}{x}$$

After integrating both sides developing our general solution.

In principle, integration factors are powerful tools for solving ordinary differential equations. In practice of this technique, they are only found in certain cases. Integrating factors can be found in important situations where $\vec{u}$ is a given function of only one variable [1]. Integrating factors are used to solve various problems including determining exact equations [2].

BODY

Let $\vec{u}(x,t)$ and $p(x,t)$ be an unknown velocity vector and pressure in the set of real numbers, $\mathbb{R}$. These functions are defined for $x\in\mathbb{R}^n$ and time $t\geq 0$. Let $f(x,t)$ be an external force applied to a fluid element creating motion of the fluid. Let $f(x,t)$ be equal to zero. I must show there exists smooth functions $p(x,t),\vec{u}(x,t)\in\mathbb{R}^3\times[0,\infty]$ to satisfy conditions (1), (2), (3), (6) and (7) of the unsolved Navier-Stokes equation. Observe a two-dimensional partial differential equation example to be, $\frac{\partial\vec{u}}{\partial x\partial t}+[\vec{u}_0(x,t)]\vec{u}=xt^2$.

We can potentially solve one of these PDE's using a newly defined technique. The new concept implements same methodology for solving an ordinary differential equation, but with partial integration. Let $\vec{u}_0(x,t)$ be an unknown velocity vector defined for position $x \in \mathrm{R}^n$ and time $t \geq 0$.

Let an initial velocity vector $\vec{u}_0(x,t)$ be equal to $x^2 t$ and treated as a **partial integrating factor** in the given partial differential equation,

$$\frac{\partial \vec{u}}{\partial x \partial t} + [x^2 t]\vec{u} = xt^2$$

with initial conditions $\vec{u}_0(x,0) = \vec{u}_0(x) = x^2(0) = 0$. To create a general solution for this partial differential equation, let our partial integrating factor be $I = e^{\int\int [x^2 t] dxdt} = e^{\frac{1}{6}x^3 t^2}$.

We then multiply our given PDE with the developed integrating factor to obtain

$$e^{\frac{1}{6}x^3t^2} \frac{\partial \vec{u}}{\partial x \partial t} + [x^2 t] e^{\frac{1}{6}x^3t^2} \vec{u} = xt^2 e^{\frac{1}{6}x^3t^2}.$$

When we partially integrate both sides with respect to x and t, we have the visual of

$$\int\int [e^{\frac{1}{6}x^3t^2} \frac{\partial \vec{u}}{\partial x \partial t} + [x^2 t] e^{\frac{1}{6}x^3t^2} \vec{u}] = \int\int xt^2 e^{\frac{1}{6}x^3t^2} \partial x \partial t.$$

The left side of the equal sign becomes

$$e^{\frac{1}{6}x^3t^2} \vec{u} = \int\int xt^2 e^{\frac{1}{6}x^3t^2} \partial x \partial t.$$

Notice I have not integrated the right side yet for a reason. From here, another new concept is introduced. The new perception is called **partial integration by parts**. Partial integration by parts applies to and will take effect after the equal sign. The first step will be to partially integrate with respect to x.

When we implement the partial integration with respect to x first, t is the constant to look like

$$t^2 \int\int x e^{\frac{1}{6}x^3t^2} \partial x \partial t$$

Let u equal x where we take the partial derivative with respect to x giving us $\frac{\partial u}{\partial x} = 1$.

Let $\partial v = e^{\frac{1}{6}x^3t^2} \partial x \partial t$ where $v = \frac{6}{x^2 t^2} e^{\frac{1}{6}x^3t^2} \partial t$ from the *integral with exponentials*,

$$\int e^{ax} \partial x = \int e^{\frac{1}{6}x^3t^2} \partial x = \int e^{[\frac{1}{6}x^2t^2]x} \partial x = \frac{6}{x^2t^2} e^{\frac{1}{6}x^3t^2} + C$$

with respect to x. Notice the algebraic manipulation to the variable x where we partiallyintegrated the exponential, e. Let this new concept be called **partial integration with exponentials**. The formula for partial integration by parts in two-dimensions will yield, $\int\int u \frac{\partial v}{\partial x \partial t} = uv - \int\int v \frac{\partial u}{\partial x \partial t}$.

We will implement partial integration by parts once with respect to t towards obtaining a general two-dimensional equation solution. The demonstration yields,

$$\int\int u \frac{\partial v}{\partial x \partial t} = uv - \int\int v \frac{\partial u}{\partial x \partial t}$$

$$t^2 \int\int x e^{\frac{1}{6}x^3t^2} \partial x \partial t = t^2 [\frac{6x}{x^2t^2} e^{\frac{1}{6}x^3t^2} - \frac{6}{x^2t^2} \int e^{\frac{1}{6}x^3t^2} \partial t]$$

$$= \frac{6}{x} e^{\frac{1}{6}x^3t^2} - \frac{6}{x^2} \int e^{\frac{1}{6}x^3t^2} \partial t.$$

From here, lets use the existing partial integration with exponentials concept with respect to t and create,

$$\int e^{at} dt = \frac{1}{a} e^{at} + C$$

where a consists of a constant and one or more variables. Since we are in partial integration, this will bring us to our solution. Then,

$$= \frac{6}{x} e^{\frac{1}{6}x^3t^2} - \frac{6}{x^2} \int e^{[\frac{1}{6}x^3t]t} \partial t$$

$$= \frac{6}{x} e^{\frac{1}{6}x^3t^2} - \frac{6}{x^2} [\frac{6}{x^3 t} e^{\frac{1}{6}x^3t^2}]$$

$$= \frac{6}{x} e^{\frac{1}{6}x^3t^2} - \frac{36}{x^5 t} e^{\frac{1}{6}x^3t^2}.$$

Joining together the completed partial integration of both sides obtains

$$e^{\frac{1}{6}x^3t^2} \vec{u} = \frac{6}{x} e^{\frac{1}{6}x^3t^2} - \frac{36}{x^5 t} e^{\frac{1}{6}x^3t^2}.$$

Now, we can solve for our general solution from the given partial differential equation. Solving for $\vec{u}$ yields,

$$\vec{u} = \frac{\frac{6}{x}e^{\frac{1}{6}x^3t^2} - \frac{36}{x^5t}e^{\frac{1}{6}x^3t^2}}{e^{\frac{1}{6}x^3t^2}}$$

$$\vec{u}(x,t) = \frac{6}{x} - \frac{36}{x^5t}$$

where the function exists $\forall t > 0$ and $\forall x \neq 0$. Now, the divergence can be shown. Finding the partial derivative of $\vec{u}$ with respect to x gives,

$$\frac{\partial \vec{u}}{\partial x} = \frac{-6}{x^2} + \frac{180}{x^6t}.$$

The divergence yields,

$$div\vec{u} = \sum_{(x,t)=1}^{\infty} \frac{\partial \vec{u}}{\partial x} = \sum_{(x,t)=1}^{\infty} \frac{-6}{x^2} + \frac{180}{x^6t} = 0$$

where $x \in \mathbb{R}^n$, $x \neq 0$ and $t > 0$. The divergence equaling to zero explains the newly formed velocity vector field possessing an existing surface integral. It also says parts of the surface have positive and negative normal components.We can now establish a pressure function using the same new concepts. Let initial pressure $p_0(x,t)$ be equal to $2xt$ as another partial integrating factor in the given PDE,

$$\frac{\partial p}{\partial x \partial t} + [2xt]p = 2xt^2.$$

Then, our integrating factor is

$$I = e^{\iint 2xtdxdt} = e^{\frac{1}{2}x^2t^2}.$$

Next, multiplying our integrating factor into the equation gives,

$$e^{\frac{1}{2}x^2t^2}\frac{\partial p}{\partial x \partial t} + [2xte^{\frac{1}{2}x^2t^2}]p = 2xt^2e^{\frac{1}{2}x^2t^2}.$$

Now, we partially integrate both sides to visually obtain,

$$e^{\frac{1}{2}x^2t^2}p = \iint 2xt^2e^{\frac{1}{2}x^2t^2}\partial x \partial t.$$

We have come across another partial integration by parts problem on the right side of the equal sign. Let $u = x$ where $\frac{\partial u}{\partial x} = 1$. Then, let $\partial v = e^{\frac{1}{2}x^2t^2}\partial x \partial t$ where we partially integrate with respect to x receiving $v = \frac{2}{xt^2}e^{\frac{1}{2}x^2t^2}\partial t$ by the partial integration with exponentials technique,

$$\int e^{ax}\partial x = \int e^{[\frac{1}{2}xt^2]x}\partial x = \frac{2}{xt^2}e^{\frac{1}{2}x^2t^2} + C.$$

Taking out the constant t since we are implementing partial integration by parts suggests,

$$2t^2\iint xe^{\frac{1}{2}x^2t^2}\partial x\partial t = 2t^2[\frac{2x}{xt^2}e^{\frac{1}{2}x^2t^2} - \int \frac{2}{xt^2}e^{\frac{1}{2}x^2t^2}\partial t]$$

$$= \frac{4xt^2}{xt^2}e^{\frac{1}{2}x^2t^2} - \frac{4t^2}{xt^2}\int e^{\frac{1}{2}x^2t^2}\partial t$$

$$= 4e^{\frac{1}{2}x^2t^2} - \frac{4}{x}\int e^{\frac{1}{2}x^2t^2}\partial t.$$

Using the partial integration with exponentials technique with respect to t shows,

$$\int e^{at}\partial t = \int e^{[\frac{1}{2}x^2t]t}\partial t = \frac{2}{x^2t}e^{\frac{1}{2}x^2t^2} + C.$$

Now, finishing up with the partial integration by parts gives,

$$= 4e^{\frac{1}{2}x^2t^2} - \frac{4}{x}\int e^{\frac{1}{2}x^2t^2}\partial t$$

$$= 4e^{\frac{1}{2}x^2t^2} - \frac{4}{x}[\frac{2}{x^2t}e^{\frac{1}{2}x^2t^2}]$$

$$= 4e^{\frac{1}{2}x^2t^2} - \frac{8}{x^3t}e^{\frac{1}{2}x^2t^2}.$$

We bring back the left side of the equal sign, solving for our two-dimensional general solution p,

$$e^{\frac{1}{2}x^2t^2}p = 4e^{\frac{1}{2}x^2t^2} - \frac{8}{x^3t}e^{\frac{1}{2}x^2t^2}$$

$$p = \frac{4e^{\frac{1}{2}x^2t^2} - \frac{8}{x^3t}e^{\frac{1}{2}x^2t^2}}{e^{\frac{1}{2}x^2t^2}}$$

$$p(x,t) = 4 - \frac{8}{x^3t}$$

where the function exists $\forall t > 0$ and $\forall x \neq 0$.

I created a third example using another space dimension, $y \in \mathbb{R}^3$, for creating $\vec{u}(x,y,t) \in \mathbb{R}^3$ as a general solution. Let the example yield an equation such as,

$$\frac{\partial \vec{u}}{\partial x \partial y \partial t} + [2x^2y^2t]\vec{u} = xy^2t$$

where $\vec{u}_0(x,y,t) = 2x^2y^2t$ is our initial velocity vector as a partial integrating factor with initial conditions $\vec{u}_0(x,y,0) = \vec{u}_0(x,y) = 2x^2y^2(0) = 0$. Then, we have

$$I = e^{\iiint 2x^2y^2 t dxdydt} = e^{\frac{1}{9}x^3y^3t^2}$$

$$e^{\frac{1}{9}x^3y^3t^2}\frac{\partial \vec{u}}{\partial x\partial y\partial t} + [2x^2y^2te^{\frac{1}{9}x^3y^3t^2}]\vec{u} = xy^2te^{\frac{1}{9}x^3y^3t^2}.$$

In this case, we must triple integrate both sides formulating,

$$\iiint [e^{\frac{1}{9}x^3y^3t^2}\frac{\partial \vec{u}}{\partial x\partial y\partial t} + [2x^2y^2te^{\frac{1}{9}x^3y^3t^2}]\vec{u}] = \iiint xy^2te^{\frac{1}{9}x^3y^3t^2}\partial x\partial y\partial t$$

$$e^{\frac{1}{9}x^3y^3t^2}\vec{u} = \iiint xy^2te^{\frac{1}{9}x^3y^3t^2}\partial x\partial y\partial t.$$

The right side of the equal sign suggests partial integration by parts to be performed three times. The partial integration by parts case with respect to x is

$$y^2t\int xe^{\frac{1}{9}x^3y^3t^2}\partial x.$$

Let $u = x$ where $\frac{\partial u}{\partial x} = 1\partial y\partial t$. Then, let $\partial v = e^{\frac{1}{9}x^3y^3t^2}\partial x\partial y\partial t$ where the partial anti-derivative with respect to x is

$$v = \frac{9}{x^2y^3t^2}e^{\frac{1}{9}x^3y^3t^2}\partial y\partial t.$$

The formula yields,

$$y^2t\int xe^{\frac{1}{9}x^3y^3t^2}\partial x = y^2t[\frac{9}{xy^3t^2}e^{\frac{1}{9}x^3y^3t^2} - \frac{9}{x^2t^2}\iint e^{\frac{1}{9}x^3y^3t^2}\partial y\partial t]$$

$$= \frac{9y^2t}{xy^3t^2}e^{\frac{1}{9}x^3y^3t^2} - \frac{9y^2t}{x^2t^2}\iint e^{\frac{1}{9}x^3y^3t^2}\partial y\partial t$$

$$= \frac{9}{xyt}e^{\frac{1}{9}x^3y^3t^2} - \frac{9}{x^2t}\iint y^2e^{\frac{1}{9}x^3y^3t^2}\partial y\partial t.$$

Then, we compute partial integration by parts a second time by letting $u = y^2$ where $\frac{\partial u}{\partial y} = 2y$ and $\partial v = e^{\frac{1}{9}x^3y^3t^2}\partial y\partial t$ where

$$v = \frac{9}{x^3y^2t^2}e^{\frac{1}{9}x^3y^3t^2}\partial t,$$

$$= \frac{9}{xyt}e^{\frac{1}{9}x^3y^3t^2} - \frac{9}{x^2t}[\frac{9y^2}{x^3y^2t^2}e^{\frac{1}{9}x^3y^3t^2} - \int\frac{18y}{x^3y^2t^2}e^{\frac{1}{9}x^3y^3t^2}\partial t]$$

$$= \frac{9}{xyt}e^{\frac{1}{9}x^3y^3t^2} - \frac{81y^2}{x^5y^2t^3}e^{\frac{1}{9}x^3y^3t^2} + \frac{162y}{x^5y^2t^3}\int e^{\frac{1}{9}x^3y^3t^2}\partial t$$

$$= \frac{9}{xyt}e^{\frac{1}{9}x^3y^3t^2} - \frac{81y^2}{x^5y^2t^3}e^{\frac{1}{9}x^3y^3t^2} + \frac{162y}{x^5y^2}\int t^{-3}e^{\frac{1}{9}x^3y^3t^2}\partial t.$$

The third and final time will show our general multi-dimensional solution with a third performance of partial integration by parts obtaining,

$$e^{\frac{1}{9}x^3y^3t^2}\vec{u} = \frac{9}{xyt}e^{\frac{1}{9}x^3y^3t^2} - \frac{81y^2}{x^5y^2t^3}e^{\frac{1}{9}x^3y^3t^2} + \frac{162y}{x^5y^2}[\frac{9}{x^3y^3t^4}e^{\frac{1}{9}x^3y^3t^2} + \frac{27}{x^3y^3t^5}e^{\frac{1}{9}x^3y^3t^2}]$$

$$e^{\frac{1}{9}x^3y^3t^2}\vec{u} = \frac{9}{xyt}e^{\frac{1}{9}x^3y^3t^2} - \frac{81y^2}{x^5y^2t^3}e^{\frac{1}{9}x^3y^3t^2} + \frac{1458}{x^8y^4t^4}e^{\frac{1}{9}x^3y^3t^2} + \frac{4374}{x^8y^4t^5}e^{\frac{1}{9}x^3y^3t^2}$$

$$\vec{u} = \frac{\frac{9}{xyt}e^{\frac{1}{9}x^3y^3t^2} - \frac{81y^2}{x^5y^2t^3}e^{\frac{1}{9}x^3y^3t^2} + \frac{1458}{x^8y^4t^4}e^{\frac{1}{9}x^3y^3t^2} + \frac{4374}{x^8y^4t^5}e^{\frac{1}{9}x^3y^3t^2}}{e^{\frac{1}{9}x^3y^3t^2}}$$

$$\vec{u}(x,y,t) = \frac{9}{xyt} - \frac{81}{x^5t^3} + \frac{1458}{x^8y^4t^4} + \frac{4374}{x^8y^4t^5}$$

where the function exists $\forall t > 0$ and $\forall x, y \neq 0$.

RESEARCH IMPLICATIONS

The new concepts developed might be able to see application towards fluid mechanics and many other fields regarding engineering sciences. I would like to present a possibility on bounded energy of the velocity. If we want to establish an existing boundary of the velocity vector, a presentation of theory must correlate. *Fubini's Theorem* may have a correlation to bounded energy on velocity vectors.

Theorem 1 (Fubini's Theorem for Wave Theory) *If u(x,y,t) is continuous on a wave specifically in region* $\mathsf{R}^3 \times [1,\infty)$, *then*

$$\iiint_{\mathsf{R}^3} u(x,y,t)dA$$

$$= \iiint u(x,y,t)dxdydt$$

$$= \iiint u(x,y,t)dydxdt$$

where the function $\vec{u}(x,y,t)$ exists $\forall t > 0$ and $\forall x, y \neq 0$.

I am deciphering another implication with the velocity vector. Let the greek letter kappa, κ, be the curvature for some known velocity vectors $\vec{u}(x,t)$ and $\vec{u}(x,y,t)$ where $\forall t > 0$ and $\forall x, y \neq 0$ such that, $\kappa = \dfrac{|\vec{u}'(t)\times\vec{u}''(t)|}{|\vec{u}'(t)|^3}$.

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