

A Brief Pedestrian Derivation of $E = mc^2$ for the Amateur Enthusiasts

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ABSTRACT

This note presents a pedestrian derivation of $E = mc^2$ for freshmen in the physical sciences, with a method similar to that of 'handling units'. Such a picturesque derivation of the formula uses nothing but Newtonian laws of motions that do not go beyond mere definitions, together with well-known elementary physical quantities such as distance, velocity, force, momentum, and energy. The pedagogic merit of such an approach is discussed.

Keywords: Special relativity, Physics education

INTRODUCTION

The following notes are stimulated from class lectures given to students in college preparing for a major in physics and to freshman in physics, chemistry and engineering. While they constitute no original contribution to a subject that has been widely discussed, it is hoped that they may be of help to some teachers, students, and any enthusiast amateurs.

In the context of a modern physics course taken during the freshmen physics program, most instructors find themselves struggling in deriving Einstein's $E = mc^2$ equation[1] without the heavy machinery techniques of calculus that have the potential to excite and intimidate most college, pre-college and high school students and in fact learners of all ages.[2, 3] Yet, $E = mc^2$ is such a celebrated equation that capture the eyes of any enthusiast amateur who seek to understand 'how the heck this formula comes from'? Freshman physics students think that this formula is mysterious and notoriously difficult to derive and proof. Still, probably the most glorious "proof" that anyone knows of is an A-bomb. An A-bomb is built on one principle; that mass can be transformed into energy, and the formula that exactly predicts this conversion between energy and mass is $E = mc^2$. What has this celebrated formula has to do with Special Relativity? The answer is that $E = mc^2$ is derived directly from special relativity. But this poses another obstacle as the enthusiast amateur is not aware of the link between $E = mc^2$ and

the special theory of relativity, and in fact, know nothing about relativity theory and its principles except the well-known fact that 'nothing can exceed the speed of light'.

It is not often realized by physics and chemistry students during their first year, that the special theory of relativity is behind several aspects of quantum mechanics like electron spin for example. As a matter of fact, the molecular properties predicted by non-relativistic quantum physics deviate significantly from experimental results. The chemistry (and physics) of some heavy elements is influenced by relativistic effects, for example, the liquidity of mercury[4], the nobility of gold and other chemical properties of heavy metal compounds such as dipole moments, biological activity, and force constants. We should also mention that relativistic treatments and corrections are routinely implemented in standard quantum chemical software packages, without even being known by the advanced students who use them. [5] To give the student a down to earth example on the importance of relativistic effects in physics and chemistry, one may ask "why is Gold yellow (golden)?" According to special relativity, an object with a velocity very close to the speed of light exhibits time dilation and length contraction. That is exactly what happens to the electrons in Gold, their velocities are close to the speed of light so that they exhibit length contractions. This relativistic effect causes the wavelength of light absorbed to shift to blue and to reflect the opposite color which is yellow.[6]

After looking at a dozen of physics textbooks,[2] and related pedagogical papers[7-9] it strikes us that no simple derivation exist that will convince the amateur on the existence of this formula without any knowledge of calculus and special relativity. Over the years, correspondence with experienced teachers and colleagues has brought us to the conclusion that presenting the derivation in a simple way that resembles “handling units” is the most convincing, eye-catching and most importantly can be remembered and memorized. Yet, such a derivation is highly physical and capture the essence of its realm power. Since our aim is to bring a simple eye-catching proof of $E = mc^2$, no intention to present the nomenclature of the formula nor its interpretation and justification. As such, we avoid the need to delve into the “mass-energy equivalence” which states that anything having mass has an equivalent amount of energy and vice versa.[10] In Treptow[11] the reader can find a pedagogic discussion on to what extent is mass conserved in the reactions of physics and chemistry. For our needs, a mass-energy equivalence is anything having energy that exhibits a corresponding mass m given by its energy E divided by the speed of light squared c^2 . As already mentioned above, beside this mass-energy equivalence, what laymen usually associate with the $E = mc^2$ equation is that nothing can exceed the speed of light c , without even relating it to the special relativity principles. Following those “knowledge”, we present, with a method similar to that of ‘handling units’, a simple and picturesque derivation of the formula using nothing but Newtonian laws of motions, that do not go beyond mere definitions, together with well-known elementary physical quantities as: distance, velocity, force, momentum and energy. In Leary and Ingham[12] the reader can find a pedagogic derivation of $E = mc^2$, which can supplement our approach.

The next section presents our derivation followed by a subsequent summary and outlook on the pedagogic merit of this presentation.

DERIVING THE FORMULA

As explained in the introduction, the derivation given here uses a mere method which resemble that of ‘handling units’ and elementary knowledge of Newtonian laws of motion that goes nothing beyond understanding the definitions of elementary quantities. *Let us consider a body that moves with a velocity v that is very close to that of c , the speed of light, $c = [speed\ of\ light]$. All we know about the ‘speed of light’ is that nothing can exceed it, no material body can reach the speed c or go*

beyond it. Let us recall that by *momentum* we mean $[mass] \times [velocity]$ and that a *Force* acting on a body is $[mass] \times [acceleration]$. From mechanics we know that impulse is the product of the average net force acting on an object and its duration, $[impulse] = [Force] \times [time\ duration]$, and impulse applied to an object produces an equivalent change in its momentum, $[impulse] = [change\ in\ momentum]$. If a constant force applied for a *time interval* Δt on a body, the momentum of the body *changes by an amount* $[Momentum] = [Force] \times [time\ interval]$. Now, what would have happened if a constant force F will act upon our body (that travel very close to the speed of light)? From elementary mechanics we know that a *Work* is done on a body when an applied force moves it through a distance, $[Work] = [Force] \times [distance]$. Since our body travel at a speed close to c , the constant force that we assume acting on it does not really change (i.e., increase) the velocity of the body. So what does change due to the force that acts on our body? Our body has a momentum ($[mass] \times [velocity]$), and since the velocity is not changing (it is already close to c , and cannot increase further) we are forced to conclude that what does change is the mass of the body.

Let us now consider the *Energy* of the system. *Energy* is the ability to do work. The *increase of energy* of the body due to the constant force which act on it along the distance the body travel, is $[Energy] = [Force] \times [distance\ the\ body\ travel]$. Now, what is exactly the distance the body travel? Recall that $[distance] = [velocity] \times [time]$. So in *unit time*, the distance the body travels is just c , the speed of light (since in our case distance $\sim [speed\ of\ light] \times [time]$). Remarkably, we can now cast this relation between energy and force as

$$[Energy] = [Force] \times [c] \quad (1)$$

We recall that the momentum of the body is equal to the force acting on the body multiply by the time interval, and in *unit time* it is simply equal to $[momentum] = [Force]$. But what is exactly the momentum gained by the body? It is merely $m \times c$. So the expression for the force acting on the body is

$$[Force] = [mass] \times [c] \quad (2)$$

Lets us now plug in expression (2) into expression (1) and we get, *in unit time*, the relation:

$$[Energy] = [Force] \times [c] = ([mass] \times [c]) \times [c] \quad (3)$$

Simplifying the above equation, one finally gets $E = mc^2$. (4)

We should remark though that such a hand-waving derivation is for this special case, where we consider a body that travels at a speed close to c , and can not explain the generic case between energy and mass.[7, 8, 10]

SUMMARY AND OUTLOOK

Probably most teachers of physics and chemistry make use at one time or another of a method that resembles that of ‘handling units’ which prove to be the simplest, memory-less, and physically intuitive approach to introducing an equation or formula behavior, like its limit and other properties. Our approach of deriving $E = mc^2$ resemble this line of approach. If used consistently and regularly, such a method-like approach for “proofing” an equation has a great pedagogic merit, that a student may be given just one set of definition which he can be taught to look upon as shorthand statements of physical laws or as definitions of physical quantities that are valid no matter in what units one may choose to express any of the quantities involved. Such an easy proof-like method can stimulate the amateurs to a better understanding of other mysterious and notoriously-difficult-to-derive formulas.

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