

## Saha Equation for Partially Ionized Relativistic Hydrogen Plasma in Rindler Space

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### ABSTRACT

We have studied Saha equation for partially ionized relativistic hydrogen plasma in Rindler space. Following the principle of equivalence, we have obtained the abundances of neutral hydrogen atoms, hydrogen ions and the electrons in dynamic equilibrium of the photo-ionization reaction of neutral hydrogen atoms and electron capture process by hydrogen ions (de-ionization process) in relativistic hydrogen plasma in Rindler space. Following the standard technique of statistical mechanics we have investigated their variations with temperature of the plasma in the Rindler space or equivalently in some uniformly accelerated frame of reference. Hence, obtained the Saha ionization formula in a partially ionized hydrogen plasma undergoing uniform acceleration. It has been observed that the abundance of neutral hydrogen atoms decreases with the increase in temperature of the plasma, which is the usual picture, whereas it increases with the increase in the strength of uniform gravitational field. The second part of this observation shows that the binding of the electrons inside hydrogen atoms increases with the increase in the strength of gravitational field or equivalently an observer with very high acceleration will see less amount of ionized hydrogen atoms compared to inertial observer.

**Keywords:** Saha equation, Uniformly accelerated frame, Inertial frame, Rindler space, Rindler coordinates, Photo ionization, Quantum binding PACS:03.65.Ge,03.65.Pm,03.30.+p,04.q

### INTRODUCTION

It is well-known that the conventional Lorentz transformations are the space- time coordinate transformations between two inertial frames of references [1]. However, following the principle of equivalence, it is trivial to obtain the space-time transformations between a uniformly accelerated frame and an inertial frame and vice-versa in the same manner as it is done in special theory of relativity [2, 3, 4, 5, 6]. In the present scenario the flat space- time geometry is called the Rindler space. For the sake of illustration of principle of equivalence, one may state, that a reference frame undergoing an accelerated motion in absence of gravitational field is equivalent to a frame at rest in presence of a gravitational field. Therefore, in the present picture, the magnitude of the uniform acceleration is exactly equal to the strength of gravitational field. We may assume that the gravitational field is produced by a strong gravitating object. We further approximate that the gravitational field is constant within a small domain of special region. Since it is exactly equal to the uniform

acceleration of the moving frame, this is also called the local acceleration of the frame.

To study saha equation in a uniformly accelerated frame of reference or in Rindler space, we first develop a formalism with the physical concepts of principle of equivalence as discussed above, and obtain the elements of the metric tensor  $g^{\mu\nu}$ . We shall show that analogous to the Minkowski space-time metric tensor, the off-diagonal elements of  $g^{\mu\nu}$  are also zero in Rindler space. In the next step, with the conventional form of action as has been defined in special theory of relativity [1], which is invariant here also, the Lagrangian of a particle (in our study which may be a hydrogen atom or hydrogen ion or an electron) is derived from Hamilton's principle. Which further gives the momentum and energy or Hamiltonian of the particle from the standard relations of classical mechanics. Then considering a partially ionized hydrogen plasma which is a reactive mixture of neutral hydrogen atoms, hydrogen ions, electrons and photons in dynamic chemical equilibrium, we shall obtain the modified form of Saha equations when observed from a

uniformly accelerated frame of reference or in Rindler space. To the best of our knowledge, the study of relativistic version of Saha equation in Rindler space has not been reported earlier. We shall also compare our findings with the conventional results.

We have organized the article in the following manner: In the next section, for the sake of completeness, we shall give a brief review from the existing literature to obtain some of the useful relations of special theory of relativity in uniformly accelerated frame. In this section we shall also obtain the relativistic version of Hamiltonian in Rindler coordinate system. In section-3 we shall investigate photo-ionization of hydrogen atoms in partially ionized hydrogen plasma in Rindler space. The expression for the number densities are obtained in Appendix-A. To obtain the number densities, we assume for the sake of simplicity that the constituents of the reactive mixture, which is a partially ionized relativistic hydrogen plasma in Rindler space, behave classically, i.e., they obey the relativistic version of Boltzmann distribution. In section-4 we have obtained the relativistic version of Saha ionization formula in Rindler space. We have shown that the concentration of neutral hydrogen atoms decreases with the increase in temperature, which is the conventional scenario, whereas it increases with the increase in the strength of gravitational field or the magnitude of acceleration of the non-inertial frame. The gravitational field, which is a classical entity enters the quantum problem through the Rindler Hamiltonian. In the last section we give the conclusion of our findings.

**BASIC FORMALISM**

In this section, for the sake of completeness, following the references [7, 8, 9] we shall establish some of the useful formulas of special theory of relativity for a uniformly accelerated frame of reference. Before we go to the scenario of uniform acceleration of the moving frame, let us first assume that the frame S' has a rectilinear motion with uniform velocity v along x-direction with respect to some inertial frame S. Further the coordinates of an event occurred at the point P (say) is indicated by (x, y, z, t) in S-frame and with (x', y', z', t') in the frame S'. The primed and the un-primed coordinates are related by the following formulae given by

$$x' = \gamma(x - vt), y' = y, z' = z \quad \text{and} \\ t' = \gamma(t - vx) \quad \text{with } \gamma = (1 - v^2)^{-1/2} \quad (1)$$

Is the well-known Lorentz factor. Throughout this article we have followed the natural system of units, i.e., the speed of light in vacuum, c = 1 and later we put the Boltzmann constant  $k_\beta = 1$  and the Planck constant  $h = 1$ . Next consider a uniformly accelerated frame S' moving with uniform acceleration  $\alpha$  also along x-direction in S-frame. Then the Rindler coordinates are given by (see the references [7,8,9]),

$$t = \left(\frac{1}{\alpha} + x'\right) \sinh(\alpha t') \\ \text{and } x = \left(\frac{1}{\alpha} + x'\right) \cosh(\alpha t') \quad (2)$$

Hence, one can also express the inverse relations

$$t' = \frac{1}{2\alpha} \ln \left(\frac{x+t}{x-t}\right) \quad \text{and } x' \\ = (x^2 - t^2)^{1/2} - \frac{1}{\alpha} \quad (3)$$

The Rindler space-time coordinates as mentioned above are then just an accelerated frame transformation of the Minkowski metric of special relativity. The Rindler coordinate transformations change the Minkowski line element from

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad \text{to} \quad (4)$$

$$ds^2 = (1 + x')^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 \quad (5)$$

Since the motion is assumed to be rectilinear and along x-direction,  $dy' = dy$  and  $dz' = dz$ . The form of the metric tensor can then be written as

$$g^{\mu\nu} = \text{diag}((1 + \alpha x)^2, -1, -1, -1) \quad (6)$$

Since we shall deal with the accelerated frame only, we have dropped the prime symbols. Now the concept of kinematics of particle motion in special theory of relativity [1], the action integral may be written as (see also [10] and [11])

$$S = -\alpha_0 \int_a^b ds \equiv \int_a^b L dt \quad (7)$$

Then using eqns.(5) and (7) and putting  $\alpha_0 = -m_0[1]$ , where  $m_0$  is the rest mass of the particle, the Lagrangian of the particle is given by[11].

$$L = -m_0[(1 + \alpha x)^2 - v^2] \quad (8)$$

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Where  $v$  is the velocity of the particle. The momentum of the particle is then given by

$$p = m_0 v [(1 + \alpha x)^2 - v^2]^{-1/2} \quad (9)$$

Hence the Hamiltonian of the particle or the single particle energy is given by

$$H = \epsilon(p) = m_0 (1 + \alpha x) \left(1 + \frac{p^2}{m_0^2}\right)^{1/2} \quad (10)$$

This is the well-known Rindler Hamiltonian. In the next sections, using this simple form of single particle energy, we obtain the number densities for the constituents of partially ionized relativistic hydrogen plasma.

### RELATIVISTIC KINETIC THEORY FOR PARTIALLY IONIZED HYDROGEN PLASMA IN RINDLER SPACE

We assume that the constituents of the plasma are neutral hydrogen, hydrogen ions and electrons. This is a reactive mixture along with photons, which are colliding with neutral hydrogen atoms and excite or ionize them. All of these constituents are in thermodynamic equilibrium in Rindler space. The single particle energy for all the constituents, except the photons are given by eqn.(10). In this expression only the rest mass will be different for different constituents. Then following [12] the expression for number density may be written as

$$n(T, u) = \frac{gz}{(2\pi)^3} \int d^3p \exp[-\beta u(\alpha)(p^2 + m_0^2)^{1/2}] \quad (11)$$

Where  $u(\alpha) = 1 + \alpha x$  and it is further assumed that within the domain denoted by  $x$ , the gravitational field  $\alpha$  is constant. Moreover, to avoid the arbitrariness of  $\alpha$  and  $x$ , we assume  $u$  as the independent variable. Here  $\beta = \frac{1}{T}$ ,  $g$  is the degeneracy of the constituent and  $z = \exp\left(\frac{\mu}{T}\right)$  is its fugacity. The above integral over momentum can be evaluated in terms of modified Bessel function of second kind of order two. In Appendix-A we have given a brief outline of its evaluation. Then we have

$$n(T, u) = \frac{gz}{2\pi^2} T m_0^2 \frac{K_2\left(\frac{m_0 u}{T}\right)}{u} \quad (12)$$

(See the references [16,17,18])

In the above expression, if we put  $u = 1$ , i.e.,  $\alpha = 0$ , we get back the conventional form of the expression for number density. Hence, one

can very easily show that the chemical potential may be expressed as

$$\mu = T \ln \left[ \frac{n(T, u) 2\pi^2 u}{g T m_0^2} \frac{1}{K_2\left(\frac{m_0 u}{T}\right)} \right] \quad (13)$$

### RELATIVISTIC VERSION OF SAHA IONIZATION FORMULA IN RINDLER SPACE

In this section we shall study the photo disintegration of hydrogen atoms in a partially ionized relativistic hydrogen plasma in Rindler space. We assume that the constituents (neutral hydrogen, hydrogen ions and electrons) are in thermodynamic equilibrium. We further assume that because of high temperature of the plasma, electrons inside the neutral hydrogen atoms are not necessarily in the ground state. Let us assume that they are in some  $n$ th bound excited state ( $n > 1$ ) and the corresponding degeneracy is  $g_n$  (see [13] and [14] for the non-relativistic calculation. Whereas in [15] a preliminary study has been done for the Saha ionization in the Rindler space. See also [16] for a related interesting work). We start with the ionization (de- ionization or capture process), given by



Where the index  $n$  indicates that the neutral hydrogen is in the  $n$ th bound excited state. We further assume that the constituents are in dynamic chemical equilibrium. In this condition the rates for the ionization process and the de-ionization process are exactly equal. Of course the equilibrium point is a function of both temperature of the plasma and the strength of uniform gravitational field. The condition for dynamic chemical equilibrium is given by

$$\mu_{H_n} = \mu_{H^+} + \mu_e \quad (15)$$

Because of the non-conservation of photon numbers in the reactive mixture, the chemical potential of photons do not appear in the chemical equilibrium condition, i.e.,  $\mu_\gamma = 0$ . Using the expression for chemical potential as given by eqn.(13) and using the chemical equilibrium condition, given by eqn.(15) and further assuming that the rest mass of  $H_n$  atom and hydrogen ion are equal, we can write

$$R(T, u) \approx \frac{n_{H_n}}{n_{H^+} + n_e} = C \frac{u}{T} \frac{1}{K_2\left(\frac{m_e u}{T}\right)} \quad (16)$$

Where  $m_e$  is the rest mass of electron and  $C$  is a constant, a function of the degeneracies of  $H_n$  atom, hydrogen ion and electron. This is the relativistic version of Saha ionization formula.

This gives the equilibrium ratio of number density of  $H_n$  atoms and the product of the number densities of hydrogen ions and the electrons. With  $u = 1$  or  $\alpha = 0$  we get back the conventional result for Saha ionization in the relativistic scenario in Minkowski space. In fig.(1) we have shown the variation of  $R(T, u)$  with temperature in MeV for  $u = 1$  (the solid curve) and with  $u$  for  $T = 5\text{MeV}$  (dashed curve). The solid curve of this figure shows that the ratio increases with temperature, which is the conventional scenario. Or in other words, we can say that the number density of neutral hydrogen atoms decreases with the increase in temperature. It actually indicates that the thermal ionization rate will increase with the increase in temperature of the plasma. The dashed curve indicates the variation of the ratio with  $u$ . It indicates that the electrons inside the neutral atoms become more bound in presence of strong gravitational field. The ionization of hydrogen atom in the ground state or in any one of the  $n$ th bound excited state. The kinetic (See eqn.(46) in page 49 of reference [12]). Hence, we have

$$\left[ \left( \frac{n_{H_n}}{n_H + n_e} \right) \left\{ \frac{(m_H + m_e)}{m_{H_n}} \right\}^{1/2} \frac{(g_H + g_e)}{g_{H_n}} \left( \frac{T}{2\pi u} \right)^{3/2} \right] \exp(-\epsilon_n/T) = 1 \tag{19}$$

Where  $\epsilon_n = m_{H^+} + m_e - m_{H_n}$ , the ionization potential for  $H_n$  atom. Hence, we can write

$$R(T, u) = \left( \frac{n_{H_n}}{n_H + n_e} \right) \propto (u/T)^{3/2} \exp(-\epsilon_n/T) \tag{20}$$

Which is the expression in the non-relativistic approximation. If we assume that the temperature is low enough and is constant, then the ratio  $R(T, u) \propto u^{3/2}$ . Which also indicates that with the increase of  $u$ , the density of neutral hydrogen atoms increases with the gravitational field in the Rindler space.

### CONCLUSION

For the sake of the presentations of results of our work in fig.(2) we have plotted the variation of  $R$  with the temperature  $T$ (in degree K) for various values of  $u$ . The upper curve is for  $u = 13.6$  whereas the bottom one for  $u = 1$  (i.e in absence of gravitational field  $\alpha=0$ ). The rest of the curves are for  $\alpha=3,8$  and  $10$ . Since most of the vital points of this work has already been discussed in the main text, here we shall give a very brief conclusion of our work. We have studied Saha ionization in a partially ionized relativistic hydrogen plasma in Rindler space. It has been observed that the conventional results are followed automatically if the presence of gravitational field is switched off. Further we have noticed that the electrons

energies are quite high compared to the ionization energy. The later is a few eV. Of course in the non-relativistic approximation it will appear in the expression for the ratio  $R(T, u)$ .

Lastly we would like to make non-relativistic approximation when  $r_0$  is large or temperature is low enough. We shall now show that the ionization potential will appear automatically in Saha ionization formula. Using the dynamical form of chemical equilibrium, given by eqn.(15) and without considering any approximation made previously, we have

$$\frac{n_{H_n}}{n_H + n_e} \left( \frac{m_H + m_e}{m_{H_n}} \right)^2 \frac{g_H + g_e}{g_{H_n}} T \frac{K_2\left(\frac{m_{H^+}}{T}\right) K_2\left(\frac{m_e}{T}\right)}{K_2\left(\frac{m_{H_n}}{T}\right)} = 1 \tag{17}$$

Now for large values of the arguments of the modified Bessel functions, we have

$$K_2(r_0) \approx \left( \frac{\pi}{2} \right)^{1/2} r_0^{-1/2} \exp(-r_0) \tag{18}$$

in the hydrogen atoms become more strongly bound with the increase in the strength of gravitational field in the Rindler space. As a future study we shall report the eigenvalue problem of hydrogen atom in Rindler space and show that the binding energy of electrons inside hydrogen atom increases linearly with the strength of gravitational field in Rindler space.

### APPENDIX-A

The expression for the number density is given by

$$n(T, u) = \frac{g}{(2\pi)^3} \int d^3p \, z \exp[-\beta u(\alpha)(p^2 + m_0^2)^{1/2}] \tag{21}$$

Let us substitute

$$\frac{u(\alpha)(p^2 + m_0^2)^{1/2}}{T} = r$$

Then  $p = \frac{T}{u} (r^2 - r_0^2)^{1/2}$

Where  $r_0 = \frac{m_0 u}{T}$ . Hence,

$$dp = \frac{T}{u} \frac{r}{(r^2 + r_0^2)^{1/2}} dr$$

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Substituting in the expression for number density, we have

$$n(T, u) = \frac{gz}{2\pi^2} \frac{T^3}{u^3} \int_{r_0}^{\infty} \exp(-r) (r^2 - r_0^2)^{1/2} r dr$$

Finally using the integral representation of  $K_\nu(r_0)$ , given by

$$K_\nu(z) = \frac{2^{\nu-1}(\nu-1)!}{(2\nu-1)!} \frac{1}{r_0^\nu} \int_{r_0}^{\infty} dr (r^2 - r_0^2)^{\nu-3/2} r \exp(-r)$$

We have (see eqn.(32), page 48 of reference [12])

$$n(T, u) = \frac{gz}{2\pi^2} \frac{T m_0^2}{u} K_2\left(\frac{m_0 u}{T}\right) \quad (22)$$

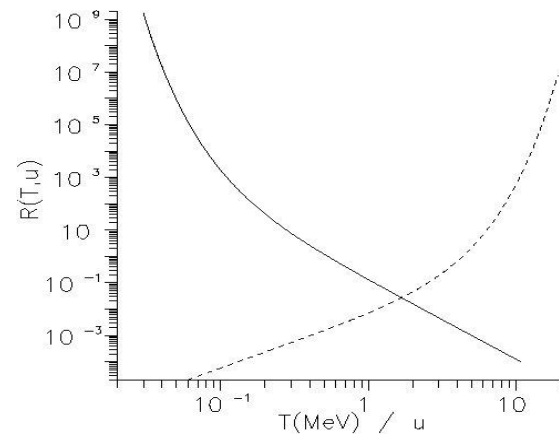
In the derivative of Saha ionization formula we shall use this expression for number density for the various constituents with different  $m_0$ . Substituting the value of fugacity  $z$ , we can express the chemical potential in terms of the number density, temperature  $T$  of the plasma and the strength of gravitational field  $u$  and is given by

$$\mu = T \left[ \ln(2\pi^2) - \ln(T) - \ln(g) - 2\ln(m_0) + \ln n - \ln K_2 u m_0 T + \ln u \right] \quad (23)$$

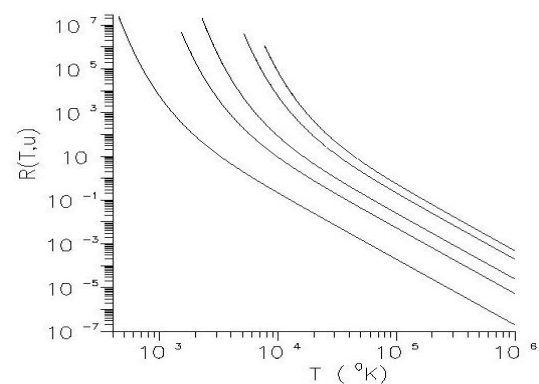
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**Figure1.** Variation of the ratio  $R(T, u)$  with  $T$  and  $u$ .



**Figure2.** The variation of  $R(T, u)$  with the temperature  $T$

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