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Double Transient Chaotic Behaviour of a Rolling Ball

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ABSTRACT

The study of relatively simple chaotic systems can provide a deep insight into the deterministic and probabilistic behaviour of the natural processes. The joint presentation of a real system and its mathematical model helps effectively to understand the intricate concepts and ideas used for the description of the physics of chaotic motion. In the present paper the dynamic behaviour of a ball moving in a complex-shaped bowl will be studied. It is shown, that this motion exhibit characteristically transient chaotic behaviour, and the boundaries of its attraction basins have typical fractal structure. Sequential magnifications of the phase space have revealed that the fractality of the basin boundaries are scale-dependent (the fractal-dimension of the basin boundaries is found to decrease and tend to unit). This behaviour has been termed the doubly transient chaos. It is an interesting fact that the character of chaos changes when a driving force is added. For example in the case of external excitation the unstable periodic orbits immediately appear, and the long term dynamics tend to permanent chaos.

Keywords: *Chaotic motion, permanent and transient chaos, fractal basinsntroduction*

INTRODUCTION

The attractor of a dynamical system is a subset of the state space to which orbits originating from typical initial conditions tend as time increases. An attractor is called chaotic or strange if it has a fractal structure in the phase space. While in case of permanent chaos the phase points of the system do never leave the chaotic attractor, if the chaos is transient the trajectory is staying only a finite time at the chaotic attractor and after it runs asymptotically into a simple attractor. If two or more attractors coexist, trajectories may hesitate for a long time before getting captured by one of the attractors. The basin of attraction of an attractor is the set of initial conditions which produces trajectory approaching the attractor. Fractal basin boundaries are common properties of dynamical systems [1]. Trajectories starting from a fractal boundary show often a transient chaos. In dissipative systems without any driving all motion must eventually cease because of the continuous decay of the energy. In this case, sequential magnifications of the phase space indicate that the set of long lifetimes becomes increasingly sparse at sufficiently small scales, so we can find that the fractality of the basin boundary is scale-dependent (the fractal-dimension of the basin boundaries is found to decrease with magnification and tend to unit). This behaviour has been termed the doubly transient chaos. It is an interesting fact that the character of chaos changes when driving is added. For example in the case of external excitation unstable periodic orbits immediately appear, and the long term dynamics tend to permanent chaos [2].

THE MECHANICAL MODEL

In our previous works we have investigated the chaotic properties of simple mechanical systems [3],[4]. In the present paper the dynamic behaviour of a ball moving in a complex-shaped vessel will be studied [5],[6]. The shape of the bowl is defined by a height function z(x,y) of the points of the bowl, this function can be identified with the gravitational potential for the moving ball. To approach the real motion the equations are completed with a term of friction. If this term is zero, the motion is conservative; in other cases it is dissipative. It is worth mentioning that if the z(x,y) function is given, then the bowl can be fabricated by a rapid prototyping procedure and the real motion of the ball can be also studied.

The motion of a point-like body of unit mass which is moving in a V(x,y) potential field under the influence of friction which is proportional with the velocity can be described by the equations:

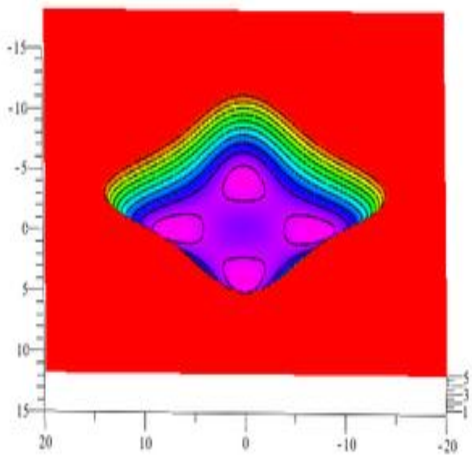
$$\ddot{x} = -\frac{\partial V}{\partial x} - \alpha\dot{x}, \quad \ddot{y} = -\frac{\partial V}{\partial y} - \alpha\dot{y} \ , \qquad (1)$$

where α is the coefficient of friction. (If α=0 then the motion is conservative.) Introducing the coordinates of the velocity as new variables, the equations can be transformed into the usual form:

$$\left.\begin{aligned} \dot{x} &= f_1(x,y,u,v) = u \ ; \\ \dot{u} &= f_2(x,y,u,v) = -\frac{\partial V}{\partial x} - \alpha u \\ \dot{y} &= f_3(x,y,u,v) = v \ ; \\ \dot{v} &= f_4(x,y,u,v) = -\frac{\partial V}{\partial y} - \alpha v \end{aligned}\right\} . \qquad (2)$$

The height (potential) function of the bowl is (x, y and z are measured in cm):

$$z(x,y) = V(x,y) = 10^{-4}\left(x^4 + 9y^4 + 26x^2y^2 - 100x^2 - 300y^2 + 5000\right) . \qquad (3)$$



1.a. *Maple display*

1.b. *A real bowl which is like the simulated one*

Figure1. A complex-shaped vessel

It is an important fact that the deepest points of the bowl situate near the four vertices of the bowl. The potential energy of the moving ball is the lowest at these points therefore these are the stable equilibrium positions of the ball.These points are: (x1=0; y1=4.0825), (x2=0; y2=-4.0825), (x3=7.0711; y3=0) (x4=-7.0711; y4=0).

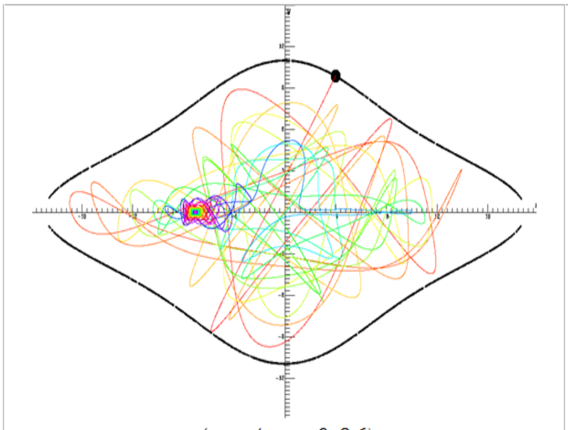
The solution of the equations of motion are studied by the Dynamics Solver [7] program which is an ideal (and free!) tool for the investigation of dynamic systems.

RESULTS

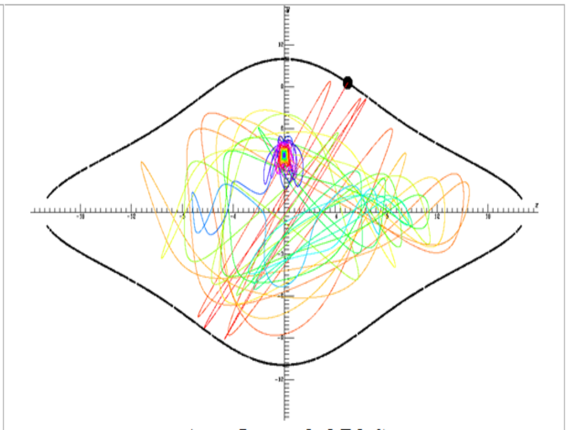
Motion with Friction

In case of frictional motion the ball comes to rest in a well near one vertex of the bowl, so as it is expected the trajectories after a transient terminate either of the minimum points of the potential. These transient motion can be often chaotic. However, in spite of the motion is initially chaotic, finally it becomes periodic or stops, so chaotic part is a transient one. In our case the attractors of the motion are the four potential well where the ball comes to a rest. The following figures show the trajectories of the motion with frictional coefficient of α=0.005. The ball was released from two different points of the rim of the bowl with zero initial velocity.

In the following the structure of the basins of attraction for the motion occurring in the bowl (in potentials) which can be seen in Fig. 1 will be revealed. As it has been mentioned the potential characterising the bowl has four potential well which were determined earlier. In the maps of the fig 2 shows below the potential wells were marked with different colours and their basins of attractions was painted by the same colour too. The colours belonging to the attractors (x1=0; y1=4.0825), (x2=0; y2=-4.0825), (x3=7.0711; y3=0), and (x4=-7.0711; y4=0) are red, green, blue and yellow, respectively.

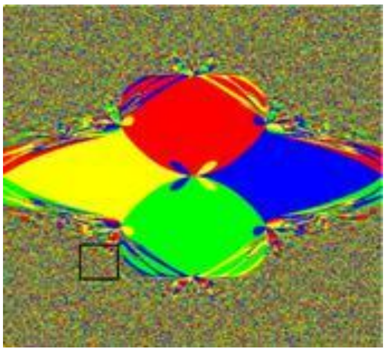


$(x_0=4;y_0=9.86)$

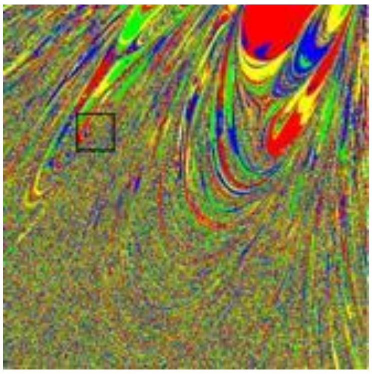


$(x_0=5;y_0=9.2796)$

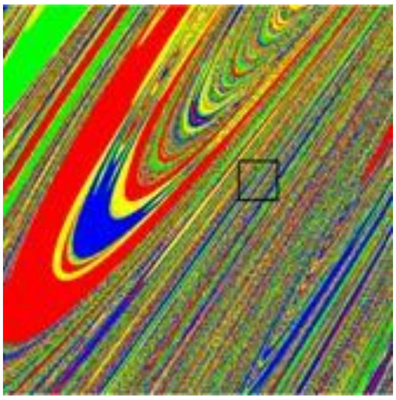
Figure2. *Trajectories of the motion with nonzero friction at different initial conditions*



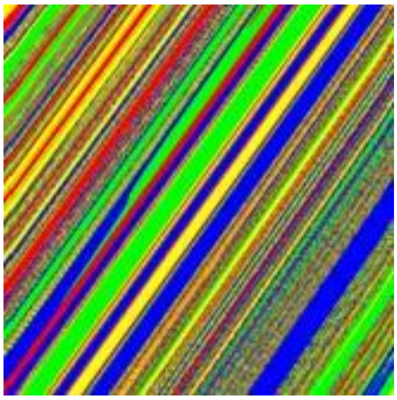
(a) $x, y \in [-10;10]$



(b) $x \in [-6;-4], y \in [-6;-4]$



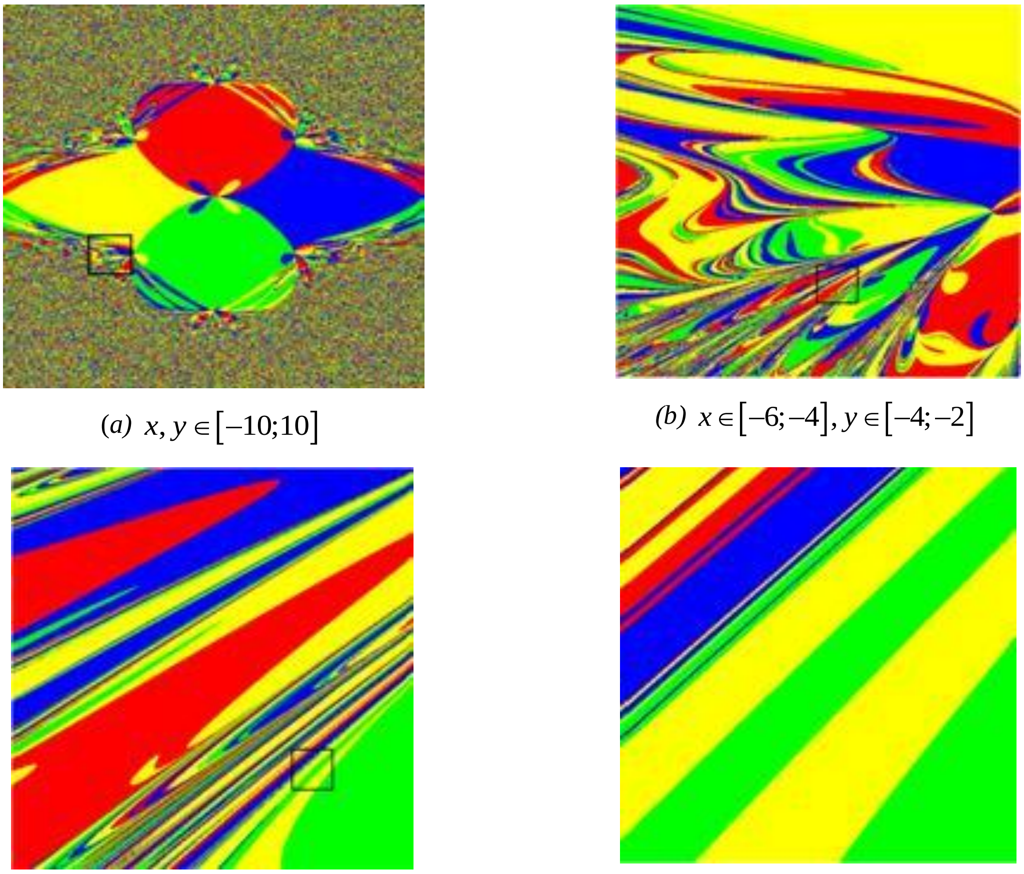
(c) $x \in [-5.6;-5.4], y \in [-4.8;-4.6]$



(d) $x \in [-5.48;-5.46], y \in [-4.7;-4.68]$

Figure 3. *Basins of attraction for the bowl of potential (3) (Friction coefficient:* $\alpha = 0.01$*, resolution:* 500×500 *initial velocity: 0)*

Figures show the structure of the basins which was formed if the friction coefficient was α=0.01. The sequence of pictures is arranged alphabetically in the figures. Every picture consists of 500×500 points and in every one a small square is chosen at a boundary of either basins. Every member of the sequence of the pictures shows the ten times magnified image of the square marked in the preceding picture.



(a) $x, y \in [-10;10]$

(b) $x \in [-6;-4], y \in [-4;-2]$

(c) $x \in [-5;-4.8], y \in [-3.6;-3.4]$

(d) $x \in [-4.86;-4.84], y \in [-3.56;-3.54]$

Figure4. *Basins of attraction for the bowl of potential (3) (Friction coefficient:* $\alpha = 0.01$, *resolution:* 500×500 *initial velocity: 0)*

The boundaries of these basins show fractal geometry which can be described by a very complicated structure like a Cantor set. In other words, whenever two basins seem to meet, we discover upon closer examination that a third basin can be found between them, and so ad infinitum.

In tables below fractal dimension of boundaries between the attraction basins shown in Fig. 3.(c) and Fig.4.(c) are given as a function of the magnification.

Table1. *Fractal dimension of boundaries between the attraction basins are given as a function of the Magnification*

	$x \in [-5;-4.8], y \in [-3.6;-3.4]$	$x \in [-5.6;-5.4], y \in [-4.8;-4.6]$
50×50	1.30	1.55
100×100	1.26	1.51
200×200	1.23	1.44
400×400	1.19	1.33
800×800	1.16	1.29

Fractals in an abstract space are sometimes not expressive and meaningful to students. In contrast to this the attraction basins of the attractors of transient chaos exhibit in the real space, so the properties of the fractals being visualized in the real geometric space appear more suggestively for the students.. The development of the fractal boundaries are illustrated well by the video [8] showing the real and simulated motion of a magnetic pendulum.

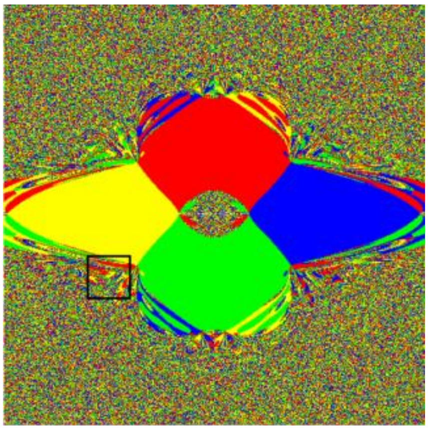
The fractal basin boundaries have shown irregular behaviour: in this case the classical parameters used to describe chaos became time dependent and the structure of the basins was not fully invariant upon magnification. The measured dimension of the basin boundaries can be non-integer over all finite scales, but have asymptotic fractal co-dimension: one. This phenomenon is recently referred as doubly transient chaos [9].

Dissipative motion with external driving force

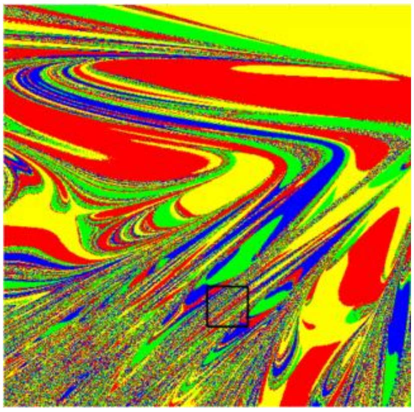
It is an interesting and important fact that the character of chaos changes when driving is added. Let's put into the system investigated a point like body (e.g. a small magnet) which attracts the moving (steel) ball. Oscillate the attracting body harmonically along the z axle, about the z_0 point with period T_z and amplitudeza. The form of the driving force is:

$$\overrightarrow{F_d}(t) = -k\frac{\left(\vec{r}_{ball}(t)-\vec{r}_{magnet}(t)\right)}{\left|\vec{r}_{ball}(t)-\vec{r}_{magnet}(t)\right|^3}\text{ , where }\vec{r}_{magnet}(t)=\left(0\ ;\ 0\ ;\ z_0+z_a\cdot\sin\left(\frac{2\pi}{T_z}t\right)\right) \quad , \qquad (4)$$

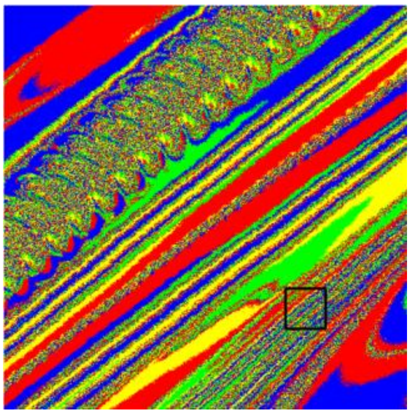
and the coupling constant acting between the magnet and the moving ball is denoted by k. For the simulations the z_0=0, z_a=0.5, T_z=0.1 values were chosen.



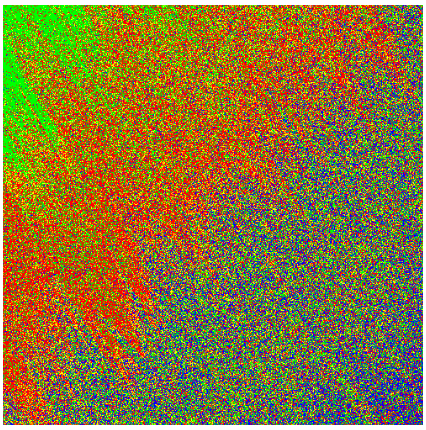
(a) $x, y \in [-10;10]$



(b) $x \in [-6;-4], y \in [-4;-2]$



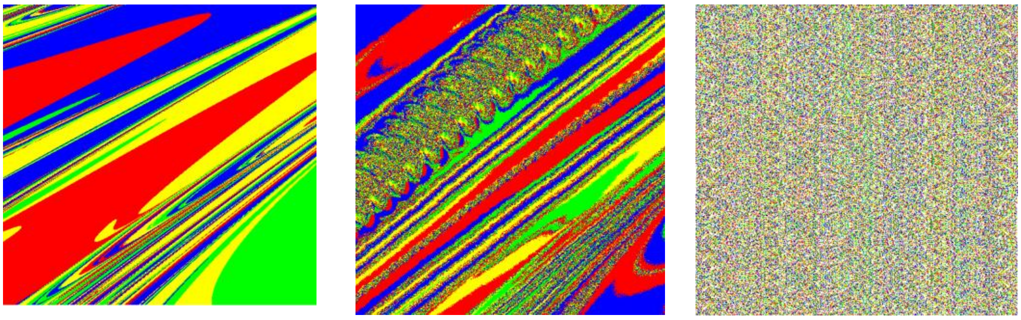
(c) $x \in [-5;-4.8], y \in [-3.6;-3.4]$



(d) $x \in [-4.86;-4.84], y \in [-3.56;-3.54]$

Fig ure5. *Basins of attraction for the bowl of potential (3) (Friction coefficient:* $\alpha = 0.01$, *and coupling constant k=0.1; resolution:* 500×500 *initial velocity: 0)*

Simulations demonstrate, that in case of small coupling constants the transient chaotic nature of the motion is maintained. However, with increasing k the characteristic time of the transient behaviour is also increasing and when k reaches a critical value (in our case k=1.1) permanent chaos appears. Studying image-sequences Fig.4 and 5 the structure of the attraction basin of the same region of the phase space can be compared at k=0 (pure dissipative case) and k=0.1. It is also visually well perceptible the increase of the fractal dimension of the boundary region with the increase of k. Fig. 6 demonstrates the change of the fractal geometry in a small region of the phase space. In case of k=0, k=0.1 and k=1.1 the fractal dimensions are 1.26, 1.79 and 2, respectively. The last dimension shows that if the coupling constant reaches its critical value, the attracting basins cease and transient chaos is replaced by permanent one.



4.a. *k=0* ***4.b.*** *k=0.1* ***4.c.*** *k=1.1*

Figure6. *Basins of attraction in the region of* $x \in [-5;-4.8]$, $y \in [-3.6;-3.4]$ *with different k.*

SUMMARY

Studying the transient chaos is very important since real physical processes reach their stationary state through transients. At university courses these transients has been neglected very often, however, the development of the stable motion might depend very sensitively on the initial part of the motion. Chaotic transient support the recognition of the importance of the initial values also. In this paper we demonstrated the importance of the transients with a simple example. It has been shown that the initial motion of a ball in a complex shaped vessel can exhibit double transient chaos, but applying a driving force the character of the motion changed to be permanent chaos.

ACKNOWLEDGMENTS

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